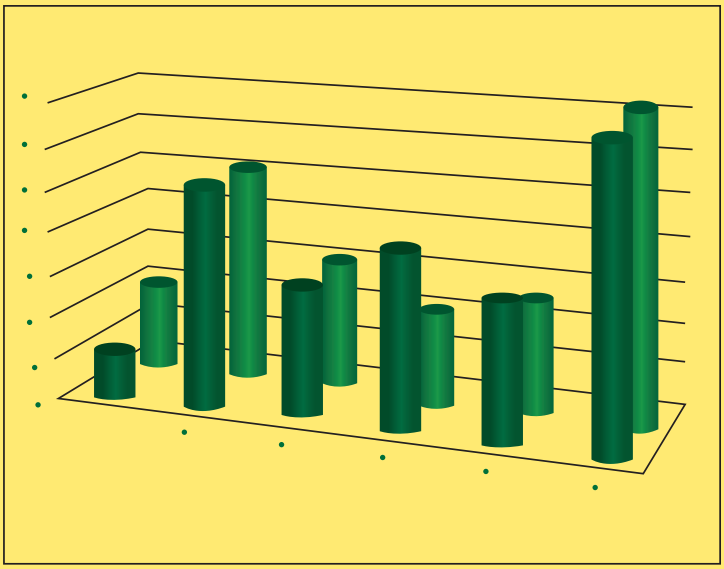
المعهد العربي للتدريب والبحوث الإحصائية

مجلة
العلوم الإحصائية



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شروط النشر في مجلة العلوم الإحصائية

1 – تنشر المجلة البحوث والدراسات العلمية في المجالات الإحصائية والمعلوماتية المكتوبة باللغة العربية والانكليزية والفرنسية على أن لا يكون البحث المقدم للنشر قد نشر او قدم للنشر في مجلات او دوريات أخرى او قدم ونشر في دوريات لمؤتمرات أو ندوات.

2 – ترسل البحوث والدراسات الى أمين التحرير على أن تتضمن اسم الباحث او الباحثين وألقابهم العلمية وأماكن عملهم مع ذكر عنوان المراسلة وأرقام الهواتف والبريد الالكتروني. وان يرسل البحث المراد نشره الكترونيا (على قرص او بالبريد الالكتروني) وفق المواصفات أدناه:

أ – أن يكون مطبوعاً على ورق حجم A4 وان يكون على شكل عمود واحد ويستخدم للغة العربية نوع حرف (Simplified Arabic) و(Times New Roman) للإنجليزية والفرنسية وبحجم خط (12). وباستخدام Microsoft Word وعلى وجه واحد للورقة.

ب – الهامش مسافة 2.5 سم لجميع جوانب الورقة.

ج – يرفق الباحث ملخصاً عن بحثه باللغتين العربية والانجليزية والفرنسية بما لا يزيد عن صفحة واحدة.

د - يتم الإشارة الى المصادر العلمية في متن البحث وفي نهايته، مع مراعاة أن لا يتضمن البحث سوى المصادر التي تم الإشارة إليها في المتن ووفق الأصول المعتمدة في ذلك (اسم المؤلف، سنة النشر، عنوان المصدر، دار النشر، البلد).

هـ– ترقم الجداول والرسوم التوضيحية وغيرها حسب ورودها في البحث، كما توثق المستعارة منها بالمصادر الأصلية.

و– أن لا يزيد عدد صفحات البحث اوالدراسة عن (25) صفحة.

3 – يتم إشعار الباحث باستلام بحثه خلال مدة لا تتجاوز يومين عمل من تاريخ استلام البحث.

4 – تخضع كافة البحوث المرسلة الى المجلة للتقييم العلمي الموضوعي ويبلغ الباحث بالتقييم والتعديلات المقترحة إن وجدت خلال مدة لا تتجاوز اسبوعان من تاريخ استلام البحث.

5 – لهيئة تحرير المجلة الحق في قبول او رفض البحث ولها الحق في إجراء أي تعديل او إعادة صياغة جزئية للمواد المقدمة للنشر بما يتماشى والنسق المعتمد في النشر لديها بعد موافقة الباحث.

6 – يصبح البحث المنشور ملكاً للمجلة ولا يجوز إعادة نشره في أماكن أخرى.

7 – تعبر المواد المنشورة بالمجلة عن آراء أصحابها، ولا تعكس وجهة نظر المجلة او المعهد العربي للتدريب والبحوث الإحصائية.

8 – ترسل البحوث على العنوان الالكتروني للمجلة:

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The Type II Generalized Topp-Leone Burr Type III Distribution: Properties and Estimation

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The Type II Generalized Topp-Leone Burr Type III Distribution: Properties and Estimation

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Abstract
In this paper, a proposed distribution called the Type II generalized Topp-Leone Burr Type III is introduced. This four-parameter distribution is obtained using Type II generalized Topp-Leone family as a generator and Burr Type III as a baseline distribution. The plots of hazard rate function of this distribution have various shapes where it can be unimodal, bimodal, decreasing or right-skewed. The main statistical properties are derived which is the reliability function, hazard rate function, reversed hazard function, cumulative hazard rate function, quantile function, median, central and non-central moments, moment generating function, order statistics and record values. The mode is obtained numerically with various parameters values. Maximum likelihood estimation for the parameters, rf and hrf of the TIIGTL-BIII distribution based on Type II censoring samples is presented. A simulation study is carried out to assess the precision of the estimators. Through an application on COVID-19 data in Canada, the superiority of the Type II generalized Topp-Leone Burr Type III distribution over some distributions is proved.
Keywords: Topp-Leone distribution, Type II generalized Topp-Leone family, Burr Type III distribution, Type II generalized Topp-Leone Burr Type III distribution, moments, order statistics, record values, maximum likelihood estimation, simulation study.

توزيع توب-ليون المعمم من النوع الثانى بير من النوع الثالث: الخصائص والتقدير

الملخص:
فى هذا البحث تم اقتراح توزيع جديد يسمى توب-ليون المعمم من النوع الثانى بير من النوع الثالث، تم إشتقاق هذا التوزيع ذو الأربع معالم بإستخدام عائلة توب-ليون المعمم من النوع الثانى كمولد وتوزيع بير من النوع الثالث كتوزيع أساسى. دالة معدل الفشل لهذا التوزيع لها عدة أشكال: أحادية المنوال أو ثنائية المنوال أو تناقصية أو ملتوية ناحية اليمين. تمت دراسة بعض الخصائص الإحصائية الأساسية لهذا التوزيع وهى: دالة البقاء، دالة معدل الفشل، الدالة المقابلة لدالة معدل الفشل، دالة معدل الفشل التراكمية، الدالة الكمية، الوسيط، العزوم المركزية واللامركزية، الدالة المولدة للعزوم، الإحصاءات الترتيبية، القيم المسجلة وتم إيجاد المنوال عددياً لقيم مختلفة من المعالم، وقد تم تقدير المعالم ودالتي البقاء والفشل للتوزيع المقترح فى حالة النوع الثانى من عينات المراقبة، وقدمت دراسة محاكاة لتقييم أداء مقدرات الإمكان الأعظم فى تقدير معالم التوزيع المقترح، وتم اثبات أفضلية توزيع توب-ليون المعمم

من النوع الثانى بير من النوع الثالث على بعض التوزيعات فى ملائمة بيانات حقيقية لكوفيد- ١٩ فى كندا.

1. Introduction

The *Topp-Leone* (TL) distribution was introduced by **Topp and Leone (1955)** as an alternative model failure data. It is a continuous unimodal distribution with a shape parameter δ as well as a bounded support $(0 < x < 1)$. Also, it is attractive as a generator. The *probability density function* (pdf) of TL is J-shaped and the *hazard rate function* (hrf) is bathtub-shaped for all values of $\delta \in [0, 1]$. These unusual characteristics are especially important in reliability applications in many fields such as economics, biology, ecology, etc., it also attracted various researchers as an alternative to beta distribution. Where in contrast to beta distribution, the TL distribution provides closed forms of the *cumulative distribution function* (cdf).

A random variable X is distributed as the TL with parameter δ denoted by $X \sim TL\ (\delta)$ if its cdf and pdf are given, respectively, by

$$F_{TL}(x;\delta) = x^{\delta}(2-x)^{\delta}, \qquad 0 < x < 1\ ;\ \delta > 0, \qquad (1)$$

$$f_{TL}(x;\delta) = 2\delta x^{\delta-1}(1-x)(2-x)^{\delta-1},\ \ 0 < x < 1;\ \delta > \qquad (2)$$

where δ is a shape parameter.

Generated families of continuous distributions are a new development for extending the classical distributions by adding new parameter(s) to a baseline distribution in order to enhance the goodness-of-fit of the distributions, and also for determining skewness and tail properties. Some of well-known families are the *Marshall-Olkin generated* (MO-G) family by **Marshall and Olkin (1997)**, Weibull-G by **Bourguignon *et al.* (2014)**, Lomax-G by **Cordeiro *et al.* (2014)**, Topp-Leone-G family by **Al-Shomrani *et al.* (2016)**, Lindley-G by **Cakmakyapan and Ozel (2017)**, among others.

TL family of distributions has received the attention of many researchers to generate new distributions and families. For instance, **Helmy (2019)** introduced Topp–Leone Pareto Type I distribution, **Oguntunde *et al*. (2019)** derived the *Topp–Leone Lomax* (TLLo) distribution, **Rasheed (2020)** provided Topp-Leone Dagum distribution, **Chipepa *et al*. (2021)** obtained the Burr III-Topp-Leone-G family, **Oluyede *et al*. (2022)** proposed the Topp-Leone Gompertz-G family, **Watthanawisut *et al*. (2022)** introduced the Beta Topp-Leone generated family, and **Nyamajiwa *et al.* (2024)** proposed the *Gamma Type II Half Logistic Topp-Leone generated* (GTIIHLTL-G) family of distributions.

Moreover, some researchers proposed different forms of generalizations of TL-G family which depend on TL distribution as a generator such as **Elgarhy *et al.* (2018)** derived the *Type II Topp-Leone generated* (TIITL-G) family, **Hassan *et al.* (2019)** suggested *Type II Generalized Topp-Leone-G* (TIIGTL-G) family, **Bantan *et al.* (2020)** presented Type II Power Topp-Leone generated family, and **Hassan *et al.* (2021)** introduced New Topp Leone-G family.

In this research, TIIGTL-G family is used to generate a new distribution called the *Type II generalized Topp-Leone Burr Type III* (TIIGTL-BIII) distribution.

Ristić and Balakrishnan (2012) suggested a gamma-G family using the gamma distribution. The cdf of this generator is defined by

$$F_{Gamma-G}(x;\omega,\gamma)=1-\frac{1}{\Gamma(\omega)}\int_0^{-logG(x;\gamma)}x^{\omega-1}e^{-x}dx,\ \ x\in\mathbb{R};\quad \omega>0. \quad (3)$$

Where $\Gamma(\omega)=\int_0^\infty x^{\delta-1}e^{-x}dx$ is the gamma function, ω is a shape parameter, and $G(x;\gamma)$ is the cdf of any continuous distribution which depends on a parameter vector γ.

Hassan *et al*. (2019) introduced a new generated family of continuous distributions with two shape parameters called *Type II generalized Topp-Leone-G* (TIIGTL-G) family, using the cdf of gamma generator that is obtained in Equation (3) with taking TL distribution as a generator and the upper bound of integration to be $1-[G(x;\gamma)]^\beta$.

Thus, the cdf and pdf of TIIGTL-G family are given, respectively, by:

$$\begin{aligned}F_{TIIGTL-G}(x;\delta,\beta,\gamma)&=1-\int_0^{1-[G(x;\gamma)]^\beta}2\delta x^{\delta-1}(1-x)(2-x)^{\delta-1}dx\\&=1-\left(1-[G(x;\gamma)]^{2\beta}\right)^\delta,\qquad x\in\mathbb{R};\quad \delta,\beta>0,\quad (4)\end{aligned}$$

$$f_{TIIGTL-G}(x;\delta,\beta,\gamma)=2\delta\beta g(x;\gamma)[G(x;\gamma)]^{2\beta-1}\left(1-[G(x;\gamma)]^{2\beta}\right)^{\delta-1},$$
$$x\in\mathbb{R};\quad \delta,\beta>0.\quad (5)$$

Where δ and β are two shape parameters and $G(x;\gamma)$ is a baseline cdf. The TIIGTL-G family can fit different data with various shapes where its density and hazard rate function accommodate different shapes which make it more flexible

Burr (1942) proposed twelve distributions as Burr family based on the Pearson differential equations for fitting cumulative frequency functions on frequency data. Its pdf has many shapes that make them applicable to a lot of applications. From the Burr family, the *Burr III* (BIII) distribution is widely used, where it has many applications in survival analysis, financial studies, reliability, and statistical quality control, among others. The BIII distribution is the inverse model of Burr XII distribution. A generalization of BIII model, named Dagum distribution **Dagum (1977)**.

A random variable X has BIII distribution if its cdf and pdf are as follows:

$$F_{BIII}(x;\nu,\lambda) = (1+x^{-\nu})^{-\lambda}, \qquad x>0; \quad \nu,\lambda>0, \tag{6}$$

$$f_{BIII}(x;\nu,\lambda) = \nu\lambda x^{-\nu-1}(1+x^{-\nu})^{-\lambda-1}, \quad x>0; \quad \nu,\lambda>0. \tag{7}$$

Where ν and λ are the shape parameters of the distribution.

This paper is organized as follows: in Section 2, The TIIGTL-BIII distribution is derived. The graphical description of the pdf and hrf of the TIIGTL-BIII distribution is presented in Section 3. The expansion of pdf of TIIGTL-BIII distribution is given in Section 4. In Section 5, the properties of the TIIGTL-BIII are obtained. In Section 6, MLEs of the parameters, rf and hrf of the TIIGTL-BIII distribution are derived based on Type II censoring samples. In section 7, a simulation study is conducted. Finally, an application on COVID-19 data in Canada is obtained in section 8.

2. The Type II Generalized Topp-Leone Burr Type III Distribution

In this section, a new distribution called TIIGTL-BIII distribution with the parameter vector $\underline{\theta} = (\delta,\beta,\nu,\lambda)$ is generated using TIIGTL-G family. The cdf of TIIGTL-BIII distribution can be derived by substituting Equation (6) into Equation (4), then it will be as follows:

$$F(x;\underline{\theta}) = 1-\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta}, \; x>0\,; \; \underline{\theta}>0. \tag{8}$$

And by substituting Equations (6) and (7) into Equation (5), the pdf of the TIIGTL-BIII distribution is given as:

$$f(x;\underline{\theta}) = 2\delta\beta\nu\lambda x^{-\nu-1}(1+x^{-\nu})^{-2\beta\lambda-1}\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta-1}, \quad x>0\,; \; \underline{\theta}>0. \tag{9}$$

where $\underline{\theta}$ are shape parameters of the distribution.

The corresponding rf and hrf of the TIIGTL-BIII distribution, respectively, are:

$$R(x;\underline{\theta}) = \left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta}, \; x>0\,; \; \underline{\theta}>0. \tag{10}$$

$$h(x;\underline{\theta}) = 2\delta\beta\nu\lambda x^{-\theta-1}(1+x^{-\nu})^{-2\beta\lambda-1}\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{-1}, \quad x>0\,; \; \underline{\theta}>0. \tag{11}$$

3. Graphical Description

In this section, the graphical description of pdf and hrf are presented which show the flexibility of the proposed model due to the different shapes of them which make this distribution fit different sets of data.

The plots of the pdf and hrf of the TIIGTL-BIII distribution are showed in Figures (1) and (2), respectively, as:

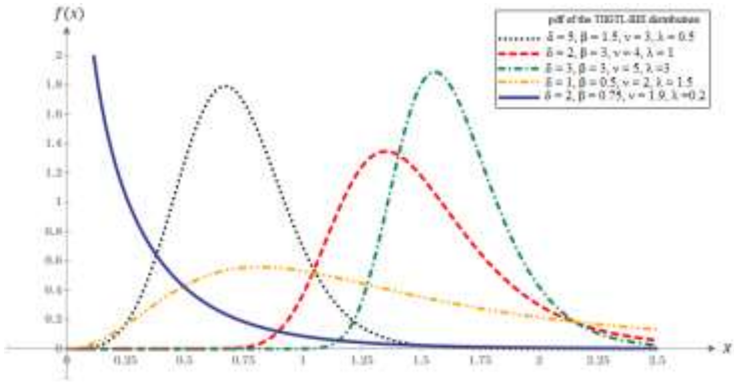

Figure (1) Plots of the pdf of TIIGTL-BIII distribution at different values of the parameters.

Figure (1) shows shapes of the pdf of TIIGTL-BIII distribution for some values of the parameters. It can be unimodal, decreasing or right (positively) skewed.

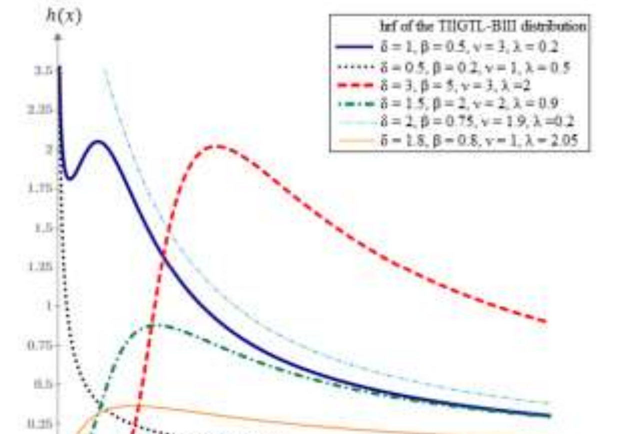

Figure (2) Plots of the hrf of TIIGTL-BIII distribution at different values of the parameters.

Figure (2) displays the hrf of TIIGTL-BIII distribution. It shows that the hrf has many shapes which are decreasing, unimodal, bimodal and right-skewed. Thus, TIIGTL-BIII distribution is flexible to use in several fields.

4. The Expansion of Probability Density Function

By using the binomial expansion $(1-y)^d = \sum_{i=0}^{\infty}(-1)^i \binom{d}{i} y^i$, then substituting into (9). The pdf can be obtained as follows:

$$f(x;\underline{\theta}) = \sum_{i=0}^{\infty} \eta_i x^{-\nu-1}(1+x^{-\nu})^{-2\beta\lambda(i+1)-1}, \quad (12)$$

where $\eta_i = 2\delta\beta\nu\lambda(-1)^i \binom{\delta-1}{i}$.

5. Statistical Properties

5.1 The Reversed Hazard Rate Function

The *reversed hazard rate function* (rhrf) of the TIIGTL-BIII distribution is obtained as follows:

$$rh(x;\underline{\theta}) = \frac{f(x;\underline{\theta})}{F(x;\underline{\theta})} = \frac{2\delta\beta\nu\lambda x^{-\nu-1}(1+x^{-\nu})^{-2\beta\lambda-1}\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta-1}}{1-\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta}}. \quad (13)$$

5.2 The Cumulative Hazard Rate Function:

The *cumulative hazard rate function* (chrf) of the TIIGTL-BIII distribution is given as:

$$H(x;\underline{\theta}) = -\ln R(x;\underline{\theta}) = -\delta \ln\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]. \quad (14)$$

5.3 The Quantile Function

The quantile function of TIIGTL-BIII distribution is derived through inverting the cdf of the TIIGTL-BIII distribution as follows:

$$F(x;\underline{\theta}) = q,$$

hence, the quantile function of the TIIGTL-BIII distribution is:

$$x_q = \left[\left(1-[1-q]^{\frac{1}{\delta}}\right)^{\frac{-1}{2\beta\lambda}} - 1\right]^{\frac{-1}{\nu}}, \quad 0 < q < 1. \quad (15)$$

The first quartile $x_{0.25}$, the median $x_{0.5}$ and the third quartile $x_{0.75}$ are given by putting $q = 0.25$, $q = 0.5$ and $q = 0.75$, respectively, into (15) as follows:

$$x_{0.25} = \left[\left(1-0.75^{\frac{1}{\alpha}}\right)^{\frac{-1}{2\beta\lambda}} - 1\right]^{\frac{-1}{\theta}}, \quad (16)$$

$$x_{0.5} = \left[\left(1-0.5^{\frac{1}{\alpha}}\right)^{\frac{-1}{2\beta\lambda}} - 1\right]^{\frac{-1}{\theta}}, \quad (17)$$

and $$x_{0.75} = \left[\left(1-0.25^{\frac{1}{\alpha}}\right)^{\frac{-1}{2\beta\lambda}} - 1\right]^{\frac{-1}{\theta}}. \quad (18)$$

Further, semi-interquartile range (quartile deviation) is calculated as:

$$Q.D = \frac{x_{0.75}-x_{0.25}}{2} = \frac{\left[\left(1-0.25^{\frac{1}{\alpha}}\right)^{\frac{-1}{2\beta\lambda}}-1\right]^{\frac{-1}{\theta}}-\left[\left(1-0.75^{\frac{1}{\alpha}}\right)^{\frac{-1}{2\beta\lambda}}-1\right]^{\frac{-1}{\theta}}}{2}. \quad (19)$$

5.4 The Mode

The mode of TIIGTL-BIII distribution is given by differentiating the pdf in (9) with respect to x and equating it to zero as follows:

$$\acute{f}(x;\underline{\theta}) = 0.$$

Then,

$$\acute{f}(x;\underline{\theta}) = 2\delta\beta\nu\lambda x^{-2\nu-2}(1+x^{-\nu})^{-4\beta\lambda-2}\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta-2} \times \left\{\nu(2\beta\lambda+1)(1+x^{-\nu})^{2\beta\lambda}\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]-(\nu+1)x^{\nu}(1+x^{-\nu})^{2\beta\lambda+1}\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]-2\beta\lambda\nu(\delta-1)\right\}, \quad (20)$$

Equation (20) is solved numerically using R program (version 4.1.1). The numerical values of the mode will be as in Table (1).

Table (1) The mode of TIIGTL-BIII for different parameter values.

$\boldsymbol{\delta}$	$\boldsymbol{\beta}$	$\boldsymbol{\nu}$	$\boldsymbol{\lambda}$	**Mode**
5	1.5	3	0.5	0.6330
2	3	4	1	1.2930
3	3	5	3	1.5275

5.5 The Moments

5.5.1 The Moments about Origin (Non-Central Moments)

The r^{th} moment about origin of TIIGTL-BIII distribution is given by:

$$\acute{\mu}_r = E(x^r) = \int_{\Omega x} x^r f(x;\underline{\theta})dx$$

$$= \int_0^{\infty} x^r.2\delta\beta\nu\lambda x^{-\nu-1}(1+x^{-\nu})^{-2\beta\lambda-1}\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta-1}dx,$$

By using integration by substitution, hence:

$$\acute{\mu}_r = \delta\int_0^1 \left[y^{\frac{-1}{2\beta\lambda}}-1\right]^{\frac{-r}{\nu}}[1-y]^{\delta-1}dy,$$

Using the negative binomial expansion: $(1-q)^{-r} = \sum_{x=0}^{\infty}\binom{r+x-1}{x}q^x$,

Thus:

$$\acute{\mu}_r = \delta(-1)^{\frac{-r}{\nu}}\sum_{k=0}^{\infty}\binom{\frac{r}{\delta}+k-1}{k}\mathrm{B}\left(1-\frac{k}{2\beta\lambda},\delta\right),\ r = 1,2,3,\ldots, \quad (21)$$

Where $\mathrm{B}(.,.)$ is the beta function and $0 < \frac{k}{2\beta\lambda} < 1$.

Then, by substituting r = 1, 2, 3, 4 into (21), the first four moments about origin can be obtained as follows:

$$\acute{\mu}_1 = \delta(-1)^{\frac{-1}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{1}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right), \quad (22)$$

$$\acute{\mu}_2 = \delta(-1)^{\frac{-2}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{2}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right), \quad (23)$$

$$\acute{\mu}_3 = \delta(-1)^{\frac{-3}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{3}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right), \quad (24)$$

$$\acute{\mu}_4 = \delta(-1)^{\frac{-4}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{4}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right). \quad (25)$$

5.5.2 The Moments about Mean (Central Moments)

The r^{th} moment about mean of TIIGTL-BIII distribution is obtained by:

$$\mu_r = E(x-\mu)^r,$$

by substituting r = 2, the variance of TIIGTL-BIII distribution $V(x)$ is:

$$V(x) = \mu_2 = E(x-\mu)^2 = \acute{\mu}_2 - \acute{\mu}_1^{\,2} = \delta(-1)^{\frac{-2}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{2}{\nu}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right) - \delta(-1)^{\frac{-1}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{1}{\nu}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right), \quad (26)$$

The third central moment is:

$$\mu_3 = \acute{\mu}_3 - 3\acute{\mu}_2\acute{\mu}_1 + 2\acute{\mu}_1^{\,3} = \delta(-1)^{\frac{-3}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{3}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right) - 3\left[\delta(-1)^{\frac{-2}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{2}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right)\right]\left[\delta(-1)^{\frac{-1}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{1}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right)\right] + 2\left[\delta(-1)^{\frac{-1}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{1}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right)\right]^3, \quad (27)$$

And the fourth central moment is:

$$\mu_4 = \acute{\mu}_4 - 4\acute{\mu}_3\acute{\mu}_1 + 6\acute{\mu}_2\acute{\mu}_1^{\,2} - 3\acute{\mu}_1^{\,4} = \delta \sum_{k=0}^{\infty} \binom{\frac{4}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right) - 4\left[\delta(-1)^{\frac{-3}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{3}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right)\right]\left[\delta(-1)^{\frac{-1}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{1}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right)\right] + 6\left[\delta(-1)^{\frac{-2}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{2}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right)\right] \times \left[\delta(-1)^{\frac{-1}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{1}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right)\right]^2 - \left[\delta(-1)^{\frac{-1}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{1}{\delta}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right)\right]^4, \quad (28)$$

The *skewness coefficient* (SC) is given as follows:

$$\alpha_3 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}}, \quad (29)$$

by substituting (26) and (27) into (29), The SC of TIIGTL-BIII distribution is:

$$\alpha_3 = \frac{\acute{\mu}_4 - 4\acute{\mu}_3\acute{\mu}_1 + 6\acute{\mu}_2\acute{\mu}_1^{\,2} - 3\acute{\mu}_1^{\,4}}{\left(\delta(-1)^{\frac{-2}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{2}{\nu}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right) - \delta(-1)^{\frac{-1}{\nu}} \sum_{k=0}^{\infty} \binom{\frac{1}{\nu}+k-1}{k} \mathrm{B}\left(1-\frac{k}{2\beta\lambda}, \delta\right)\right)^{\frac{3}{2}}}. \quad (30)$$

The *kurtosis coefficient* (KC) is given as:

$$\alpha_4 = \frac{\mu_4}{\mu_2^{\,2}}, \quad (31)$$

by substituting (26) and (28) into (31), The KC of TIIGTL-BIII distribution is:

$$\alpha_4 = \frac{\acute{\mu}_4 - 4\acute{\mu}_3\acute{\mu}_1 + 6\acute{\mu}_2{\acute{\mu}_1}^2 - 3{\acute{\mu}_1}^4}{\left((-1)^{\frac{-2}{\nu}}\sum_{k=0}^{\infty}\binom{\frac{2}{\nu}+k-1}{k}\mathrm{B}\left(1-\frac{k}{2\beta\lambda},\delta\right) - \delta(-1)^{\frac{-1}{\nu}}\sum_{k=0}^{\infty}\binom{\frac{1}{\nu}+k-1}{k}\mathrm{B}\left(1-\frac{k}{2\beta\lambda},\delta\right)\right)^2}. \qquad (32)$$

5.6 The Moment Generating Function

The moment generating function of TIIGTL-BIII distribution $M_x(t)$ is obtained by:

$$M_x(t) = E(e^{tx})$$

$$= \int_0^{\infty} e^{tx}.\, 2\delta\beta\nu\lambda x^{-\theta-1}(1+x^{-\nu})^{-2\beta\lambda-1}\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta-1}dx,$$

using integration by substitution, hence:

$$M_x(t) = \delta\int_0^1 e^{t\left(y^{\frac{-1}{2\beta\lambda}}-1\right)^{\frac{-1}{\nu}}}(1-y)^{\delta-1}dy,$$

By using the power series for the exponential function and the negative binomial expansion, thus:

$$M_x(t) = \delta\sum_{r=0}^{\infty}\frac{t^r}{r!}(-1)^{\frac{-r}{\nu}}\sum_{k=0}^{\infty}\binom{\frac{r}{\nu}+k-1}{k}\beta\left(1-\frac{k}{2\beta\lambda},\delta\right), \qquad (33)$$

Where $0 < \frac{k}{2\beta\lambda} < 1$, and as noted in Equation (33) $M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\acute{\mu}_r$.

5.7 The Order Statistics

Let $x_1, x_2 \dots, x_r, \dots, x_n$ are *independent identical distributed* (i.i.d.) random variables from TIIGTL-BIII distribution and the $x_{(1)} \le x_{(2)} \dots \le x_{(r)} \le \cdots \le x_{(n)}$ is the corresponding order statistics, hence the pdf of the rth order statistic is obtained by:

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x) = \frac{n!}{(r-1)!(n-r)!}\left[1-\left(1-[1+x^{-\nu}]^{-2\beta\lambda}\right)^{\delta}\right]^{r-1}\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta(n-r)} \times 2\delta\beta\nu\lambda x^{-\nu-1}(1+x^{-\nu})^{-2\beta\lambda-1}\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta-1}, x_{(r)} > 0. \qquad (34)$$

When r = 1, the smallest order statistics is:

$$f_{1:n}(x) = n[1-F(x)]^{n-1}f(x) = 2n\delta\beta\nu\lambda x^{-\nu-1}(1+x^{-\nu})^{-2\beta\lambda-1}\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{n\delta-1}, \;\; x_{(1)} > 0. \qquad (35)$$

When r = n, the largest order statistics is:

$$f_{n:n}(x) = n[F(x)]^{n-1}f(x)$$

$$= 2n\delta\beta\nu\lambda x^{-\nu-1}(1+x^{-\nu})^{-2\beta\lambda-1}\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta-1}$$

$$\times\left[\left(1-[1-(1+x^{-\nu})]^{-2\beta\lambda}\right)^{\delta}\right]^{n-1}, \qquad x_{(n)} > 0. \qquad (36)$$

5.8 The Record Values

Consider sequence of i.i.d. random variables from TIIGTL-BIII distribution, where $x_{U(n)} = max\{x_1, x_2 \dots, x_r, \dots, x_n\}$ is an upper record value and $x_{L(n)} = min\{x_1, x_2 \dots, x_r, \dots, x_n\}$ is a lower record value.

The pdf of the n[th] upper record value is obtained as follows:

$$f_{U(n)}(x) = \frac{1}{\Gamma n}(-\ln[1-F(x)])^{n-1} f(x) = \frac{1}{\Gamma \mathrm{n}}\left[-\ln\left(\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta}\right)\right]^{n-1} . 2\delta\beta\nu\lambda x^{-\nu-1}(1+x^{-\nu})^{-2\beta\lambda-1} \times \left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta-1}, \quad x > 0 \tag{37}$$

The pdf of lower record value is given as:

$$f_{L(n)}(x) = \frac{1}{\Gamma n}[-\ln F(x)]^{n-1} f(x) = \frac{1}{\Gamma \mathrm{n}}\left[-\ln\left(1-\left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta}\right)\right]^{n-1} . 2\delta\beta\nu\lambda x^{-\nu-1}(1+x^{-\nu})^{-2\beta\lambda-1} \times \left[1-(1+x^{-\nu})^{-2\beta\lambda}\right]^{\delta-1}, \quad x > 0 \tag{38}$$

6. Maximum Likelihood Estimation

Assuming that $x_{(1)} \le x_{(2)} \le \dots \le x_{(r)}$ is a censored sample of size r from TIIGTL-BIII distribution with parameter vector $\underline{\theta} = (\delta, \beta, \nu, \lambda)$, then the likelihood function of Type II censored sample is given as follows:

$$L(\underline{\theta}; \underline{x}) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} f(x_{(1)}; \underline{\theta})\right]\left[R(x_{(r)}; \underline{\theta})\right]^{n-r} = \frac{n!}{(n-r)!}(2\delta\beta\nu\lambda)^r \left[\prod_{i=1}^{r} x_{(1)}^{-(\nu+1)}\right]\left[\prod_{i=1}^{r}\left(1+x_{(i)}^{-(\nu)}\right)^{(-2\beta\lambda+1)}\right] \times \left[\prod_{i=1}^{r}\left(1-\left[1+x_{(i)}^{-\nu}\right]^{-2\beta\lambda}\right)^{\delta-1}\right]\left[1-\left(1+x_{(r)}^{-\nu}\right)^{-2\beta\lambda}\right]^{(n-r)\delta}. \tag{39}$$

Since this likelihood function becomes the likelihood function in case of complete sample when r = n.

Then, the natural logarithm of the likelihood function is obtained as follows:

$$\ell = \ln L(\underline{\theta}; \underline{x}) = \ln\frac{n!}{(n-r)!} + r\ln(2\delta\beta\nu\lambda) - (\nu+1)\sum_{i=1}^{r}\ln(x_{(i)}) - (2\beta\lambda+1)\sum_{i=1}^{r}\ln\left[1+x_{(i)}^{-\nu}\right] + (\delta-1)\sum_{i=1}^{r}\ln\left[1-\left(1+x_{(i)}^{-\nu}\right)^{-2\beta\lambda}\right] + (n-r)\delta\ln\left[1-\left(1+x_{(r)}^{-\nu}\right)^{-2\beta\lambda}\right]. \tag{40}$$

For a random variable x has TIIGTL-BIII distribution subject to the regularity conditions (where x does not depend on $\underline{\theta}$), the point estimators of $\underline{\theta}$ of TIIGTL-BIII distribution can be derived by obtaining the differentiation of ℓ in (40) with respect to $\underline{\theta}$ as follows:

$$\frac{\partial\ell}{\partial\delta} = \frac{r}{\delta} + \sum_{i=1}^{r}\ln\left[1-\left(1+x_{(i)}^{-\nu}\right)^{-2\beta\lambda}\right] + (n-r)\ln\left[1-\left(1+x_{(r)}^{-\nu}\right)^{-2\beta\lambda}\right] \tag{41}$$

$$\frac{\partial \ell}{\partial \beta}=\frac{r}{\beta}-2\lambda\sum_{i=1}^{r}\ln\left(1+x_{(i)}^{-\nu}\right)+2\lambda(\delta-1)\sum_{i=1}^{r}\frac{\left(1+x_{(i)}^{-\nu}\right)^{-2\beta\lambda}\ln\left(1+x_{(i)}^{-\nu}\right)}{1-\left(1+x_{(i)}^{-\nu}\right)^{-2\beta\lambda}}$$

$$+2\delta\lambda(n-r)\frac{\left(1+x_{(r)}^{-\nu}\right)^{-2\beta\lambda}\ln\left(1+x_{(i)}^{-\nu}\right)}{1-\left(1+x_{(r)}^{-\nu}\right)^{-2\beta\lambda}}, \tag{42}$$

$$\frac{\partial \ell}{\partial \nu}=\frac{r}{\nu}-\sum_{i=1}^{r}\ln\left(x_{(i)}\right)+(2\beta\lambda+1)\sum_{i=1}^{r}\frac{x_{(i)}^{-\theta}\ln\left(x_{(i)}\right)}{1+x^{-\theta}}\quad-2\beta\lambda(\delta-$$

$$1)\sum_{i=1}^{r}\frac{x_{(i)}^{-\theta}\ln\left(x_{(i)}\right)\left(1+x_{(i)}^{-\nu}\right)^{-2\beta\lambda-1}}{1-\left(1+x_{(i)}^{-\nu}\right)^{-2\beta\lambda}}-(n-r)2\delta\beta\lambda\frac{x_{(r)}^{-\theta}\ln\left(x_{(r)}\right)\left(1+x_{(r)}^{-\nu}\right)^{-2\beta\lambda-1}}{1-\left(1+x_{(r)}^{-\nu}\right)^{-2\beta\lambda}}, \tag{43}$$

And

$$\frac{\partial \ell}{\partial \lambda}=\frac{r}{\lambda}-2\beta\sum_{i=1}^{r}\ln\left(1+x_{(i)}^{-\nu}\right)+2\beta(\delta-1)\sum_{i=1}^{r}\frac{\left(1+x_{(i)}^{-\nu}\right)^{-2\beta\lambda}\ln(1+x_{(i)}^{-\nu})}{1-\left(1+x_{(i)}^{-\nu}\right)^{-2\beta\lambda}}$$

$$+2\beta\delta(n-r)\frac{\left(1+x_{(r)}^{-\nu}\right)^{-2\beta\lambda}\ln(1+x_{(r)}^{-\nu})}{1-\left(1+x_{(r)}^{-\nu}\right)^{-2\beta\lambda}}. \tag{44}$$

The ML estimator of the parameter δ can be obtained by equating Equation (41) to zero and solving, then it will be as follows:

$$\hat{\delta}=-\frac{r}{\sum_{i=1}^{r}ln\left[1-\left(1+x_{(i)}^{-\nu}\right)^{-2\beta\lambda}\right]+(n-r)\,ln\left[1-\left(1+x_{(r)}^{-\nu}\right)^{-2\beta\lambda}\right]}. \tag{45}$$

The ML estimates of $\underline{\theta}$ are given by equating Equations (41) – (44) to zero and solving numerically.

Based on invariance property of the ML estimators (MLEs), The MLEs of $R\left(x;\underline{\theta}\right)$ and $h\left(x;\underline{\theta}\right)$ can be obtained, respectively, by:

$$\hat{R}\left(x;\underline{\hat{\theta}}\right)=\left[1-\left(1+x^{-\hat{\nu}}\right)^{-2\hat{\beta}\hat{\lambda}}\right]^{\hat{\delta}},\qquad x>0, \tag{46}$$

And

$$\hat{h}\left(x;\underline{\hat{\theta}}\right)=2\hat{\delta}\hat{\beta}\hat{\nu}\hat{\lambda}x^{-\hat{\nu}-1}\left(1+x^{-\hat{\nu}}\right)^{-2\hat{\beta}\hat{\lambda}-1}\left[1-\left(1+x^{-\hat{\nu}}\right)^{-2\hat{\beta}\hat{\lambda}}\right]^{-1},\ x>0 \tag{47}$$

7. Simulation Study

In this section, a simulation study is conducted to evaluate the performance of the ML estimators of the parameters, rf and hrf for TIIGTL-BIII distribution based on Type II censoring scheme using Mathematica 12 by the following steps:

1- Generating random samples of sizes (n = 30, 60, 100) based on two levels of censoring [0% (complete sample) and 60%] from the TIIGTL-BIII distribution using the transformation between the uniform and TIIGTL-BIII distributions (the inverse function) which is as follows:

$$x=\left[\left(1-[1-u]^{\frac{1}{\delta}}\right)^{\frac{-1}{2\beta\lambda}}-1\right]^{\frac{-1}{\nu}},\qquad 0<u<1,$$

With assuming that the population parameter values are:

(δ = 1, β = 0.5, ν = 2, λ = 1.5).

2- The ML estimates of the parameters are given by equating Equations (41) – (44) to zero and solving numerically by using initial values of the parameters (δ = 1.3, β = 1, ν = 1.7, λ = 1.2) then substituting these estimates into Equations (45) and (46) to obtain the ML estimates of the rf and hrf at time $x_0 = 0.5$.
3- Repeating this process m = 1000 times.
4- The ML estimates are calculated by averaging over the m repetitions.
5- The *mean square error* (MSE), bias, variance, *lower limit* (LL) and *upper limit* (UL) for 95% *confidence intervals* (CIs) for the parameters, rf and hrf are computed according to the following equations:

$$MSE = \frac{\sum_{i=1}^{m}\left(estimated\ value_{(i)} - true\ value\right)^2}{m},$$

$$Bias = E(estimator) - true\ value,$$

$$var = MSE - (Bias)^2,$$

$$\text{LL} = estimate - z_{1-\frac{\alpha}{2}}\sqrt{var(estimator)},$$

$$\text{UL} = estimate + z_{1-\frac{\alpha}{2}}\sqrt{var(estimator)},$$

And Length = UL - LL.

Then, the numerical results are obtained in the Tables (2) - (5).

Table (2) The ML estimates, MSE, variance, bias and 95% confidence intervals of parameters of the TIIGTL-BIII distribution for different sample sizes in case of complete sample (0% censoring) and m = 1000.
(δ = 1.3, β = 1, ν = 1.7, λ = 1.2)

n	r	θ	MLE	MSE	Variance	Bias	CI		Length
							UL	LL	
30	30	δ	1.3562	0.1710	0.0441	0.3562	1.7678	0.9447	0.8231
		β	0.8866	0.1592	0.0097	0.3866	1.0801	0.6931	0.3870
		ν	1.7507	0.1099	0.0477	-0.2493	2.1787	1.3226	0.8561
		λ	1.0639	0.2042	0.0140	-0.4361	1.2961	0.8317	0.4643
60	60	δ	1.3516	0.1490	0.0253	0.3516	1.6637	1.0396	0.6240
		β	0.8789	0.1481	0.0045	0.3789	1.0107	0.7471	0.2636
		ν	1.7286	0.0992	0.0255	-0.2714	2.0416	1.4156	0.6260
		λ	1.0546	0.2048	0.0065	-0.4453	1.2128	0.8965	0.3163
100	100	δ	1.3463	0.1403	0.0204	0.3463	1.6262	1.0664	0.5598
		β	0.8754	0.1439	0.0030	0.3754	0.9827	0.7682	0.2145
		ν	1.7069	0.1063	0.0204	-0.2931	1.9867	1.4271	0.5597
		λ	1.0505	0.2046	0.0043	-0.4495	1.1792	0.9218	0.2574

Table (3) The ML estimates, MSE, variance, bias and 95% confidence intervals of the parameters of the TIIGTL-BIII distribution for different sample sizes in the case of 60% censoring and m = 1000. (δ = 1.3, β = 1, ν = 1.7, λ = 1.2)

n	r	θ	MLE	MSE	Variance	Bias	CI		Length
							UL	LL	
30	12	δ	1.4107	0.3359	0.1672	0.4107	2.2123	0.6092	1.6031
		β	0.9016	0.1936	0.0322	0.4016	1.2536	0.5497	0.7039
		ν	1.8221	0.7064	0.6748	-0.1779	3.4321	0.2121	3.2200
		λ	1.0820	0.2212	0.0464	-0.4180	1.5043	0.6596	0.8447
60	24	δ	1.3775	0.2320	0.0895	0.3775	1.9639	0.7910	1.1728
		β	0.8730	0.1581	0.0189	0.3730	1.1428	0.6033	0.5395
		ν	1.8871	0.3591	0.3464	-0.1128	3.0407	0.7336	2.3070
		λ	1.0476	0.2319	0.0273	-0.4524	1.3713	0.7239	0.6474
100	40	δ	1.3703	0.1757	0.0386	0.3703	1.7553	0.9854	0.7700
		β	0.8605	0.1414	0.0115	0.3605	1.0704	0.6506	0.4198
		ν	1.8805	0.2134	0.1991	-0.1195	2.7550	1.0060	1.7490
		λ	1.0326	0.2350	0.0165	-0.4674	1.2845	0.7807	0.5038

Concluding remarks:

Tables (2) and (3) show that:

- As the sample size increases, the ML estimates of the TIIGTL-BIII distribution become close to the population parameter values which are (δ = 1, β = 0.5, ν = 2, λ = 1.5) in most cases.
- The MSE, variance and length of CI decrease, the bias approaches to zero and the CIs become narrower, in most cases, as the sample size n increases.
- As the level of censoring decreases, the ML estimates become close to the population parameter values, in most cases.
- The MSE, variance and length of CI decrease and the bias approaches to zero, in most cases, as the level of censoring decreases [see **Ashour *et al*. (2022)**].

The ML estimates for rf and hrf are contained in Tables (4) and (5):

Table (4) The ML estimates, MSE, variance, bias and 95% confidence intervals of the rf and the hrf of the TIIGTL-BIII distribution for different sample sizes in the case of complete sample (0% censoring) and m = 1000.
(δ = 1.3, β = 1, ν = 1.7, λ = 1.2 and $x_0 = 0.5$)

n	r	rf and hrf	MLE	MSE	Variance	Bias	CI		Length
							UL	LL	
30	30	$R\left(x_0;\underline{\theta}\right)$	0.9073	0.0024	0.0024	-0.0033	1.0037	0.8108	0.1929
		$h\left(x_0;\underline{\theta}\right)$	0.4708	0.0355	0.0355	-0.0007	0.8400	0.1016	0.7385
60	60	$R\left(x_0;\underline{\theta}\right)$	0.9066	0.0012	0.0012	-0.0039	0.9733	0.8400	0.1333
		$h\left(x_0;\underline{\theta}\right)$	0.4790	0.0157	0.0157	0.0075	0.7245	0.2334	0.4910
100	100	$R\left(x_0;\underline{\theta}\right)$	0.9046	0.0008	0.0008	-0.0059	0.9591	0.8501	0.1090
		$h\left(x_0;\underline{\theta}\right)$	0.4852	0.0105	0.0103	0.0137	0.6845	0.2859	0.3985

Table (5) The ML estimates, MSE, variance, bias and 95% confidence intervals of the rf and the hrf of the TIIGTL-BIII distribution for different sample sizes in the case of 60% censoring and m = 1000.
(δ = 1.3, β = 1, ν = 1.7, λ = 1.2 and $x_0 = 0.5$)

n	r	rf and hrf	MLE	MSE	Variance	Bias	CI		Length
							UL	LL	
30	12	$R\left(x_0;\underline{\theta}\right)$	0.9043	0.0085	0.0084	-0.0062	1.0843	0.7244	0.3599
		$h\left(x_0;\underline{\theta}\right)$	0.2649	2.4704	2.4278	-0.2066	3.3189	-2.789	6.1079
60	24	$R\left(x_0;\underline{\theta}\right)$	0.9068	0.0044	0.0044	-0.0037	1.0372	0.7765	0.2606
		$h\left(x_0;\underline{\theta}\right)$	0.3889	2.1577	2.1509	-0.0826	3.2635	-2.4856	5.7490
100	40	$R\left(x_0;\underline{\theta}\right)$	0.9046	0.0020	0.0020	-0.0060	0.9916	0.8176	0.1741
		$h\left(x_0;\underline{\theta}\right)$	0.4822	0.2908	0.2908	0.0107	1.5391	-0.5747	2.1138

Concluding remarks:

From Tables (4) and (5), it is concluded that:

- As the sample size n increases, the MSE, variance and length of rf and hrf decrease, the bias approaches zero and the CIs become narrower, in most cases.
- As the level of censoring decreases, the MSE, variance and length of rf and hrf decrease and the bias approaches zero in most cases.

8. Application

In this section, an application on COVID-19 in Canada is performed to demonstrate the flexibility and applicability of the TIIGTL-BIII distribution. The TIIGTL-BIII distribution is compared with the Topp Leone-Burr III (TL-BII) and Burr III (BIII) distributions. The data set is obtained from **Almetwally *et al*. (2021)** and fitted using the three models. The pdfs of the TL-BIII and BIII distributions are as follows, respectively,

$$f_{TL-BIII}(x;\delta,\nu,\lambda) = 2\delta\nu\lambda x^{-\nu-1}(1+x^{-\nu})^{-\lambda-1}\left[1-(1+x^{-\nu})^{-\lambda}\right] \times \left[1-\left(1-[1+x^{-\nu}]^{-\lambda}\right)^{2}\right]^{\delta-1}, \quad x>0\,; \quad \delta,\beta,\nu,\lambda>0.$$

$$f_{BIII}(x;\nu,\lambda) = \nu\lambda x^{-\nu-1}(1+x^{-\nu})^{-\lambda-1}, \qquad x>0\,; \quad \nu,\lambda>0.$$

The descriptive statistics of the data set are obtained using SPSS 26. *Kolmogorov-Smirnov* (K-S) statistic and its p-value are calculated by R program (version 4.1.1). The ML estimates for the parameters, rf and the hrf and the goodness-of-fit statistics including the *-2loglikelihood* (-2$\mathcal{L}$), *Akaike information criterion* (AIC), *corrected Akaike information criterion* (CAIC) and *Bayesian information criterion* (BIC) are computed using Mathematica 12.

The lower values of K-S, -2$\mathcal{L}$, AIC, CAIC and BIC, and the highest p-value of the K-S statistic is an indicator to the better model.

The AIC, BIC and CAIC are calculated according to the following equations:

$$AIC = 2k - 2\mathcal{L},$$
$$BIC = k\ln(n) - 2\mathcal{L},$$

and

$$CAIC = AIC + \frac{2k(k+1)}{n-k-1} = -2\mathcal{L} + \frac{2kn}{n-k-1},$$

Where

k is the number of the parameters,

$\mathcal{L}$ is the natural logarithm of the likelihood function value substituted by ML estimates, and n is the sample size.

This real data set represents a COVID-19 drought mortality rate from 10 April to 15 May 2020 of 36 days belong to Canada and it is as follows:

Table (6) Data of COVID-19 in Canada.

COVID-19 data belong to Canada of 36 days	3.1091	3.3825	3.1444	3.2135	2.4946	3.5146
	4.9274	3.3769	6.8686	3.0914	4.9378	3.1091
	3.2823	3.8594	4.0480	4.1685	3.6426	3.2110
	2.8636	3.2218	2.9078	3.6346	2.7957	4.2781
	4.2202	1.5157	2.6029	3.3592	2.8349	3.1348
	2.5261	1.5806	2.7704	2.1901	2.4141	1.9048

The descriptive statistics of these data are obtained in the following table:

Table (7) the descriptive statistics of COVID-19 data in Canada.

N	Min.	Max.	Range	Mean	Standard Deviation	Mode	Skewness	Kurtosis
36	1.5157	6.8686	5.3529	3.2816	0.9985	3.1091	1.267	3.825

This table shows that the skewness measure is positive, which indicating that the distribution of the data is skewed to the right. Also, the kurtosis measure value is more than 3 therefore, the distribution of the data is leptokurtic (peaked).

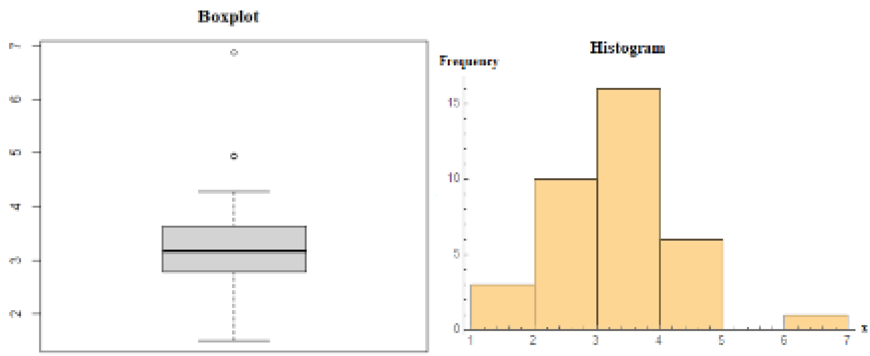

Figure (3) Boxplot and histogram for the data of COVID-19 in Canada.

In Figure (3), the boxplot indicates that the data are right-skewed and the histogram implies that they are unimodal.

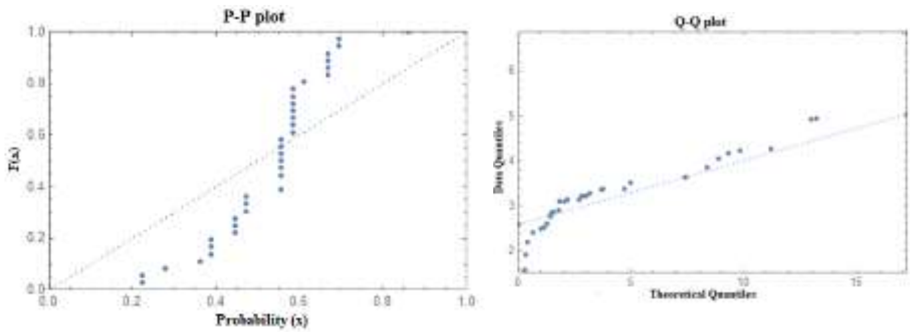

Figure (4) P-P plot and Q-Q plot for the for the TIIGTL-BIII distribution for COVID-19 data of Canada.

Figure (4) displays that the TIIGTL-BIII distribution fit the data.

Table (8) Goodness-of-fit statistics of the fitted distributions for the data of COVID-19 in Canada.

Model	K-S	p-Value	$-2\mathcal{L}$	AIC	BIC	CAIC
TIIGTL-BIII	0.2500	0.2106	104.2602	112.2602	118.5943	113.5505
TL-BIII	0.3056	0.0694	126.8713	132.8713	137.6219	133.6213
BIII	0.3611	0.0183	164.4387	168.4387	171.6058	168.8024

From Table (8), since the level of significance $\alpha = 0.01$ is less than the p-value for the three models, then they fit the data. The model with the smallest values of the goodness-of-fit statistics and a highest p-value of K–S statistic is the TIIGTL-BIII distribution. This indicates that the TIIGTL-BIII distribution fits this data set better than the TL-BIII and BIII distributions.

Table (9) MLE estimates of the fitted TIIGTL-BIII distribution for the data of COVID-19 in Canada.

$\underline{\theta}$, rf and hrf	MLE
δ	6.9360
β	2.8851
ν	1.2270
λ	1.9234
$R\left(x_0;\underline{\theta}\right)$	0.9999
$h\left(x_0;\underline{\theta}\right)$	0.00003

In Table (9), the MLEs of the parameters are quite close to the initial values of the parameters which are ($\delta = 3.15, \beta = 1.8, \nu = 0.8, \lambda = 1.2$). From this application, we can conclude that the TIIGTL-BIII distribution has flexibility in modelling the data over the TL-BIII and BIII distributions.

Conclusion

In this paper, the new distribution called TIIGTL-BIII is proposed. The plots of pdf and hrf and the main statistical properties including the reliability function, hazard rate function, reversed hazard function, cumulative hazard rate function, quantile function, median, central and non-central moments, moment generating function, order statistics, and record values are derived. The mode of the TIIGTL-BIII distribution is derived numerically with different parameter values. Estimating the parameters, rf and hrf of the TIIGTL-BIII distribution using the ML method of estimation based on Type II censoring samples is performed to evaluate the precision of the estimators. Moreover, an application for real data set of COVID-19 drought mortality rate in Canada is performed. This application proves the efficiency and flexibility of the proposed distribution in modelling the data rather than TL-BIII and BIII distributions.

References:

1. **Almetwally, E. M., Alharbi, R., Alnagar, D., & Hafez, E. H. (2021).** A new inverted topp-leone distribution: applications to the COVID-19 mortality rate in two different countries. *Axioms*, **10**(1), 25.
2. **Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S. and Shahbaz, M. Q. (2016).** Topp-Leone family of distributions: Some properties and application. *Pakistan Journal of Statistics and Operation Research*. **XII**, 3, 443-451.
3. **Ashour, E. Y., Albadawy, A. E., AL-Dayian, G. R. and EL-Helbawy, A. A. (2022).** Pak. J. Statist. 2022 Vol. 38 (3), 297-314 EXPONENTIATED INVERTED TOPP-LEONE DISTRIBUTION. *Pak. J. Statist*, **38**(3), 297-314.
4. **Bantan, R. A., Jamal, F., Chesneau, C. and Elgarhy, M. (2020).** Type II Power Topp-Leone generated family of distributions with statistical inference and applications. *Symmetry*, **12**(1), 75.
5. **Bourguignon, M., Silva, R. B. and Cordeiro, G. M. (2014).** The Weibull-G family of probability distributions. *Journal of data science*, **12**(1), 53-68.
6. **Burr, I. W. (1942).** Cumulative frequency functions. *The Annals of mathematical statistics*, **13**(2), 215-232.
7. **Cakmakyapan, S. and Ozel, G. (2017).** The Lindley family of distributions: properties and applications. *Hacettepe Journal of Mathematics and Statistics*, **46**(6), 1113-1137.
8. **Chipepa, F., Oluyede, B. and Peter, P. O. (2021).** The Burr III-Topp-Leone-G family of distributions with applications. *Heliyon*, **7**(4), e06534.
9. **Cordeiro, G. M., Ortega, E. M., Popović, B. V. and Pescim, R. R. (2014).** The Lomax generator of distributions: Properties, minification process and regression model. *Applied Mathematics and Computation*, **247**, 465-486.
10. **Dagum, C.A. (1977).** A New model of personal income distribution: specification and estimation. *Eco. Apl.*, **30**, 413-437.

11. **Elgarhy, M., Arslan Nasir, M., Jamal, F. and Ozel, G. (2018).** The type II Topp-Leone generated family of distributions: Properties and applications. *Journal of Statistics and Management Systems*, **21**(8), 1529-1551.
12. **Hassan, A. S., Elgarhy, M. and Zubair, A. (2019).** Type II Generalized Topp-Leone Family of Distributions: Properties and Applications. *Journal of Data Science*, **17**, 638-659.
13. **Hassan, A. S., El-Sherpieny, E. A. and El-Taweel, S. A. (2021).** New Topp Leone-G Family with Mathematical Properties and Applications. *Journal of Physics: Conference Series*. **1860**(1), 012011.
14. **Helmy, N. (2019).** Topp–Leone Pareto Type I Distribution. *Al-Azhar Scientific Journal of the Commercial Faculties*, **21**(1), 1-26.
15. **Marshall, A. W. and Olkin, I. (1997).** A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, **84**(3), 641-652.
16. **Nyamajiwa, V. Z., Musekwa, R. R. and Makubate, B. (2024).** Application of the New Extended Topp-Leone Distribution to Complete and Censored Data. *Revista Colombiana de Estadística*, **47**(1), 37-65.
17. **Oguntunde, P. E., Khaleel, M. A., Okagbue, H. I. and Odetunmibi, O. A. (2019).** The Topp–Leone Lomax (TLLo) distribution with applications to airbone communication transceiver dataset. *Wireless Personal Communications*, **109**(1), 349-360.
18. **Oluyede, B., Chamunorwa, S., Chipepa, F. and Alizadeh, M. (2022).** The Topp-Leone Gompertz-G family of distributions with applications. *Journal of Statistics and Management Systems*, 1-25.
19. **Rasheed, N. (2020).** Topp-Leone Dagum Distribution. *Research Journal of Mathematical and Stat*, **8**(1), 16-30.
20. **Ristić, M. M. and Balakrishnan, N. (2012).** The gamma-exponentiated exponential distribution. *Journal of statistical computation and simulation*, **82**(8), 1191-1206.
21. **Topp, C. W. and Leone, F. C. (1955).** A family of J-shaped frequency functions. *Journal of the American Statistical Association*, **50**(269), 209-219.
22. **Watthanawisut, A., Bodhisuwan, W., & Supapakorn, T. (2022).** The Beta Topp-Leone Generated Family of Distributions and Theirs Applications. *Thailand Statistician*, **20**(3), 489-489.

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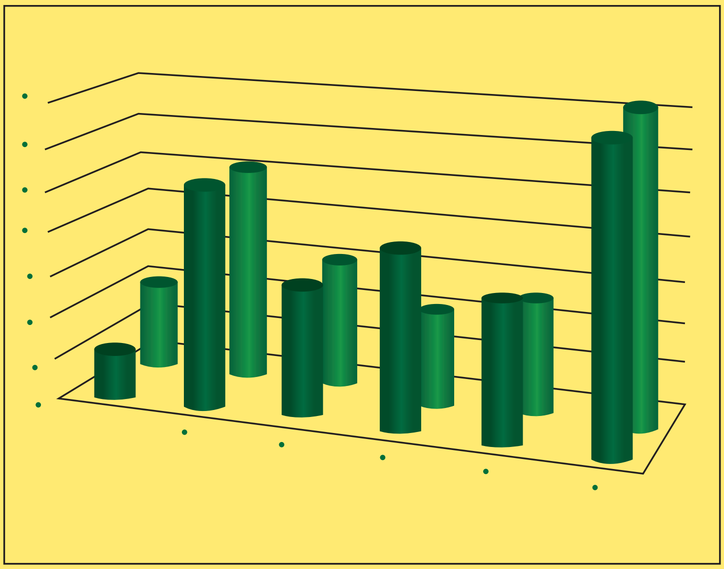

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