تأثير حجم العينة على تقدير معلمة الشكل للتوزيع الطبيعي الملتوي

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الملخص

يركز البحث على تقدير معلمة الشكل للتوزيع الطبيعي الملتوي باستخدام كل من طريقة العزوم وطريقة الإمكان الأعظم (الأكبر). ولقد تم بحث تأثير تغيير حجم العينة على المقدرات الناتجة من خلال دراسة بعض الخصائص لهذه المقدرات مثل مقدار التحيز ومتوسط مربعات الخطأ عند أحجام عينات مختلفة مثل (10،15،30،50،100) ولقد تم توليد عدد 1000 عينة تتبع هذا التوزيع وتقدير معلمة التوزيع لكل عينة. وقد وجد أن قيمة مقدار التحيز لمقدرات العزوم تنخفض مع زيادة حجم العينة، أما بالنسبة لمقدار متوسط مربعات الخطأ النسبي فإن قيمته تتردد بين الزيادة والنقصان مع زيادة حجم العينة. وتم أيضا ملاحظة أن قيم مقدرات "الإمكان الأكبر" الناتجة قريبة من القيمة المبدئية المفترضة وأن قيمة مقدار التحيز لهذه المقدرات تقل مع زيادة حجم العينة. وبالنسبة لمتوسط مربعات الخطأ النسبي فإن قيمته تقل بزيادة حجم العينة. ولقد إستخدمت الباحثة أسلوب نظام "بيرسون" للمنحنيات التكرارية لتحديد التوزيع العيني لمقدرات العزوم ومقدرات الإمكان الأكبر الناتجة، ووجد أن هذه التقديرات تتبع النوع الأول من منحنيات "بيرسون".

The Effect of Sample Size on the Estimation of the Shape Parameter of the Skewed Normal Distribution

Abstract

This paper is concerned with estimating the shape parameter of the skew normal distribution using the methods of moments and maximum likelihood. The effect of the sample size on the estimation is investigated through studying the properties of the resulting estimates using different sample sizes $(n = 10,15,30,50,100)$. Simulated data is used to satisfy this purpose. The number of replications is $N = 1000$. The bias for the moment estimates decreased as the sample size increased and the relative mean square error fluctuating as the sample size increased. The resulting maximum likelihood estimates are close to their initial values and the bias decreased as the sample size increased. In addition the relative mean square error decreased as the sample size increased. The Pearson's system for frequency curves is used to determine the sampling distribution for the resulting moments and the maximum likelihood estimates. It is found that the resulting estimates have type I .
Key words: *skew normal, moments, maximum likelihood, sampling distribution*

1. Introduction

Bayes and Branco (2007) pointed out that the skew-normal distribution includes the normal distribution as a special case. This family of distributions has a shape parameter that defines the direction of the asymmetry of the distribution, also called skewness parameter. They showed that the Bayesian approach is a good alternative to make inference under the skewness parameter. They presented a brief discuss about prior choice and proposed an approximation for the Jeffreys prior developed by Liseo and Loperfdo (2006). They also provided an approximation for the bias correction factor proposed by Sartori (2006). A simulation study is presented to compare the bias, mean square error and interval estimates using the maximum likelihood and different Bayes estimators. Finally, hypotheses tests are considered using Bayes factor and the likelihood ratio test.

Martinez *et al* (2008) proposed an alternative approach to the one in Henze (1986) for deriving the odd moments of the skew-normal distribution considered in Azzalini (1985). The approach is based on a Pascal type triangle, which seems to greatly simplify moment's computation. Moreover, it is shown that the likelihood equation for estimating the asymmetry parameter in such model is generated as orthogonal functions to the sample vector. As a consequence, conditions for a unique solution of the likelihood equation are established, which seem to hold in more general setting.

Ali, M.M. *et al* (2009) constructed some skewed distributions with probability density functions (pdf) of the form $2f(u)G(\lambda u)$ where λ is a real number, $f(.)$ is taken to be a Laplace pdf while the cumulative density function $G(.)$ comes from one of Laplace, double Weibull, reflected Pareto, reflected beta prime, or reflected generalized uniform distribution. Properties of the resulting distributions are studied. In particular, expressions for the moments of these distributions and the characteristic functions are derived. However, as some of these quantities could not be evaluated in closed forms, special functions have been used to express them. Graphical illustrations of the pdfs of the skewed distributions are also given. Further, skewness-kurtosis graphs for these distributions have been drawn.

Elal-Oliver (2010) introduced an alternative form of generate asymmetry in the normal distribution that allows to fit unimodal and bimodal data sets. Basic properties of this new distribution, such as stochastic representation, moments, maximum likelihood and the singularity of the Fisher information matrix are studied. The methodology developed is illustrated with a real application.

Azzalini and Arellano – Valle (2013) pointed out that the skew-normal and the skew-t distributions are parametric families which are currently under intense investigation since they provide a more flexible formulation compared to the classical normal and t distributions by introducing a parameter which regulates their skewness. While these families enjoy attractive formal properties from the probability viewpoint, a practical problem with their usage in applications is the possibility that the maximum likelihood estimate of the parameter which regulates skewness diverges. This situation has vanishing probability for increasing sample size, but for finite samples it occurs with non-negligible probability, and its occurrence has unpleasant effects on the inferential process. Methods

for overcoming this problem have been put forward both in the classical and in the Bayesian formulation, but their applicability is restricted to simple situations.

They formulated a proposal based on the idea of penalized likelihood, which has connections with some of the existing methods, but it applies more generally, including the multivariate case.

Lahcene (2013) proposed a new extended model of Pearson. This model associated to this heuristic is implemented to validate the result of the generalized Pearson family routine in the specific cases. This study is presented same applications of Pearson families of distributions, and gave the new extended, which extends the classical Pearson family. Various properties of this new family are investigated and then exploited to derive several related results, especially characterizations, in probability. The statistical applications of the results based on health related data are included. It is hoped that the findings of this work will be useful for the practitioners in various fields of theoretical and applied sciences.

This paper is divided into four sections. The first section contains the introduction. The moments and maximum likelihood estimates of the shape parameter of the skew normal distribution are discussed in the second section. The third section introduced the effect of the sample size on the estimation of the shape parameter of the skew normal distribution. The fourth section contains the numerical study and the main results of this paper.

2. Moments and Maximum Likelihood Estimates of the Shape Parameter of the Skew Normal Distribution

(2.1) Moments Estimate:

A random variable X is distributed skew normal distribution with the shape parameter λ if X has the density function:

$$f(x,\lambda) = 2\phi(x)\Phi(\lambda x) \quad , \quad -\infty < X < \infty \qquad (1)$$

where $\phi(x)$ and $\Phi(\lambda x)$ are the standard normal density and distribution function respectively. The probability density function (pdf) of a skew normal random variable can be expressed in the following form:

$$f(x,\lambda) = 2\frac{1}{\sqrt{2\pi}}e^{\frac{X^2}{2}}\int_{-\infty}^{\lambda X}\frac{1}{\sqrt{2\pi}}e^{\frac{t^2}{2}}\,dt \qquad (2)$$

and the cumulative distribution function is given by:

$$F(x,\lambda) = \int_{-\infty}^{x}\frac{1}{\pi}e^{\frac{X^2}{2}}\int_{-\infty}^{\lambda x}e^{\frac{t^2}{2}}\,dtdx \qquad (3)$$

Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed skew normal distribution with the shape parameter λ . The method of moments is used to obtain the estimate for the shape parameter λ by calculating the population mean for the skew normal distribution and equating the population mean with the sample mean $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$, as follows:

$$E(X) = \frac{\lambda}{\sqrt{1+\lambda^2}}\sqrt{\frac{2}{\pi}} = \frac{\sum_{i=1}^{n} X_i}{n} \qquad (4)$$

(2.2) Maximum Likelihood Estimate:

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the skew normal distribution with shape parameter λ then the pdf of a skew normal random variable is given by:

$$f(x,\lambda) = \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Phi(\lambda x) \qquad (5)$$

Then the likelihood function is given by:

$$L(x,\lambda) = \frac{2^n}{(2\pi)^{n/2}} e^{\frac{\sum_{i=1}^{n} x_i^2}{2}} \prod_{i=1}^{n} \Phi(\lambda x_i) \qquad (6)$$

The natural logarithm of the likelihood function is given by:

$$\ln(x,\lambda) = n\ln 2 - \frac{n}{2}\ln(2\pi) - \frac{\sum_{i=1}^{n} x_i^2}{2} + \sum_{i=1}^{n} \ln \Phi(\lambda x_i) \qquad (7)$$

Differentiating equation (7) with respect to λ and setting to zero we obtain:

$$\frac{d\ln(x,\lambda)}{d\lambda} = \sum_{i=1}^{n} \frac{\phi(\hat{\lambda} x_i)}{\Phi(\hat{\lambda} x_i)}(x_i) = 0 \qquad (8)$$

The nonlinear likelihood equation (8) can be solved numerically to obtain the maximum likelihood estimate (MLE) of the parameter λ .

(2.3) Pearson's system for frequency curves:

To determine the sampling distribution of the resulting estimates the value of the following criterion is calculated:

$$k = \frac{\beta_1(\beta_2+3)^2}{4(2\beta_2 - 3\beta_1 - 6)(4\beta_2 - 3\beta_1)} \qquad (9)$$

where β_1 is the measure of skewness and β_2 is the measure of kurtosis.
This criterion is called kappa measure. The value of this criterion determines the types of Pearson curves as follows:

1. If $k < 0$ this corresponds to beta distribution or type I distribution.
2. If $k > 1$ this corresponds to beta distribution of the second type or type VI distribution.
3. If $0 < k < 1$ this corresponds to type IV distribution.

3. The Evaluation of the Resulting Estimates

To study the effect of the sample size on the estimation of the shape parameter of the skew normal distribution different sample sizes are used to satisfy this purpose $(n = 10,15,30,50,100)$. Methods of moments and maximum likelihood are used to obtain the estimate of the shape parameter . The moments estimate for the parameter λ is obtained numerically by solving equation (4). The maximum likelihood estimate for the parameter λ is obtained numerically by solving equation (8).
The resulting estimates are evaluated using the following measures; the bias and the mean square error. These measures are calculated as follows:

$$\text{bias } (\hat{\lambda}) = mean(\hat{\lambda}) - \lambda$$

$$\text{relative mean square error } (\hat{\lambda}) = \frac{\sqrt{mse(\hat{\lambda})}}{\lambda} \qquad (11)$$

4. Numerical Study

The numerical results based on simulated data:

1. Given initial value λ_0 generate random samples of size $n = 10,15,30,50,100$ from skew normal distribution using the following formula:

$$z = \frac{\lambda_0}{\sqrt{1+\lambda_0^2}}|w| + \frac{1}{\sqrt{1+\lambda_0^2}}v \qquad (12)$$

where the variables have standard normal distribution with mean zero and standard deviation one (see Henze, N.(1986)).

2. For each sample size and the initial value of the shape parameter λ the moment estimate of the parameter $\boldsymbol{\lambda}$ can be obtained by solving equation (4) and the MLE by soling equation (8).
3. Repeat steps (1) and (2) N times where $N = 10000$ for $n = 10,15,30,50,100$.
4. The bias and mean square error of the resulting estimates for the moment of the parameter (λ) are computed by averaging over the N repetitions using equations (10) and (11).

   For the estimated parameter λ we assume that the actual population values are $\lambda = 1,2,4$ (small, medium and large skewness respectively).

Table (1) display the moment estimates, the bias and the mean square error for the shape parameter λ of the skew normal distribution for different sample sizes $(n = 10,15,30,50,100)$.

Table (1) Moment estimates, the bias and the mean square error for the shape parameter λ of the skew normal distribution when $(\lambda = 1,2,4)$ for the sample sizes $(n = 10,15,30,50,100)$

n	initial value λ_0	estimation	bias	Relative mean square error
10	1	0.599	−0.401	0.812
	2	0.598	−1.402	0.774
	4	0.665	−3.335	0.848
15	1	0.605	−0.395	0.818
	2	0.600	−1.4	0.781
	4	0.728	−3.272	0.832
30	1	0.545	−0.455	0.891

Table (1) continuous

n	initial value λ_0	estimation	bias	Relative mean square error
30	2	0.675	−1.325	0.744
	4	0.883	−3.117	0.788
50	1	0.445	−0.555	0.98
	2	0.709	−1.291	0.725
	4	0.943	−3.057	0.77
100	1	0.325	−0.675	1.085
	2	0.792	−1.208	0.673
	4	1.024	−2.976	0.745

From table (1) the value of the bias decreased as the sample size increased. The value of the relative mean square error fluctuating between decreasing and increasing as the sample size increased.

To study the sampling distribution of the resulting estimates the value of the criterion k is calculate $(k = -0.029)$. Since this value is negative then the random variable has type I of the Pearson's system of frequency curves. Then the probability density function of this type is as follows:

$$f(x) = y_0\left(1+\frac{x}{a_1}\right)^{m_1}\left(1+\frac{x}{a_2}\right)^{m_2} \quad ,-a_1 < X < a_2, m_1, m_2 > 0 \quad (13)$$

which is a particular case from the beta distribution of the first kind. The parameters $r, m_1, m_2, a_1, a_2, y_0$ are calculated as follows:

$$r = \frac{6(\beta_2 - \beta_1 - 1)}{6 + 3\beta_1 - 2\beta_2} = 16.261 \quad (14)$$

$$a_1 + a_2 = \frac{1}{2}\sqrt{\mu_2\{\beta_1(r+2)^2 + 16(r+1)\}} = 1.473 \quad (15)$$

$$m_1, m_2 = \frac{1}{2}\left\{r - 2 \pm r(r+2)\sqrt{\frac{\beta_1}{\beta_1(r+2)^2 + 16(r+1)}}\right\} \quad (16)$$

$$\frac{m_1}{a_1} = \frac{m_2}{a_2} \rightarrow m_1 = 5.777, m_2 = 8.484, a_1 = 0.597, a_2 = 0.876 (17)$$

$$y_0 = \frac{N}{a_1 + a_2} \frac{m_1^{m_1} m_2^{m_2}}{(m_1 + m_2)^{m_1+m_2}} \frac{\Gamma(m_1 + m_2 + 2)}{\Gamma(m_1 + 1)\Gamma(m_2 + 1)} = 32.834 (18)$$

$$Mode = Mean - \frac{1}{2} \frac{\mu_2 r + 2}{\mu_2 r - 2} = 0.659 \quad (19)$$

Figure (1) represents the sampling distribution for the moment estimates.

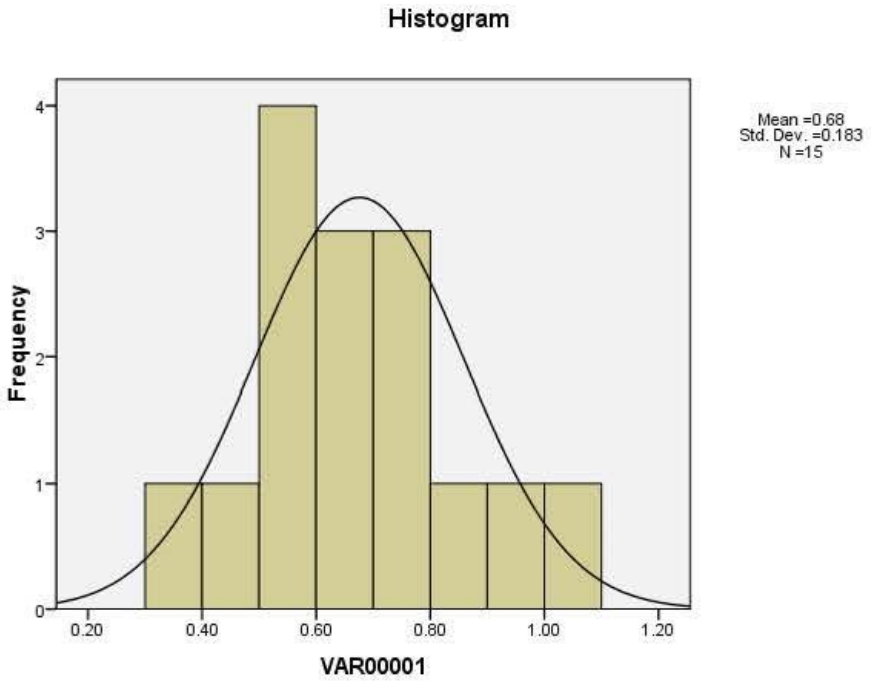

Figure (1) the sampling distribution for the moment estimates

Table (2) display the maximum likelihood estimates, the bias and relative mean square error for the shape parameter λ of the skew normal distribution when $(\lambda = 1,2,4)$ for different sample sizes $(n = 10,15,30,50,100)$.

Table (2) Maximum likelihood estimates, the bias and the mean square error for the shape parameter λ of the skew normal distribution when $(\lambda = 1,2,4)$ for the sample sizes $(n = 10,15,30,50,100)$

n	Initial value λ_0	estimation	bias	relative mean square error
10	1	4.398	3.398	3.533
	2	4.485	2.485	1.369
	4	4.488	0.488	0.304
15	1	3.611	2.611	2.676
	2	3.626	1.626	0.868

Table (2) continuous

n	Initial value λ_0	estimation	bias	relative mean square error
	4	3.642	**−0.358**	0.181
30	1	2.733	1.733	1.752
	2	2.732	0.732	0.386
	4	2.733	**−1.267**	0.323
50	1	2.309	1.309	1.315
	2	2.31	0.31	0.169
	4	2.31	**−1.69**	0.424
100	1	1.94	0.94	0.942
	2	1.94	**−0.06**	0.041
	4	1.94	**−2.06**	0.515

Results & Conclusion:

From table (4.3) the bias is decreased as the sample size is increased. In addition the relative mean square error decreased as the sample size increased.

It is found that the value of the criterion k for the vector of the resulting maximum likelihood estimates is $(k = -0.041)$. Since this value is negative then the maximum likelihood estimates has type I of the Pearson's system of frequency curves. Then the probability density function of this type is as follows:
Then the probability density function of this type is as follows:

$$f(x) = y_0\left(1+\frac{x}{a_1}\right)^{m_1}\left(1+\frac{x}{a_2}\right)^{m_2} \quad ,-a_1 < X < a_2, m_1, m_2 > 0 \quad (20)$$

which is a particular case from the beta distribution of the first kind. The parameters $r, m_1, m_2, a_1, a_2, y_0$ are calculated as follows:

$$r = \frac{6(\beta_2 - \beta_1 - 1)}{6 + 3\beta_1 - 2\beta_2} = 1.207 \quad (21)$$

$$a_1 + a_2 = \frac{1}{2}\sqrt{\mu_2\{\beta_1(r+2)^2 + 16(r+1)\}} = 2.775 \quad (22)$$

$$m_1, m_2 = \frac{1}{2}\left\{r - 2 \pm r(r+2)\sqrt{\frac{\beta_1}{\beta_1(r+2)^2 + 16(r+1)}}\right\} \quad (23)$$

$$\frac{m_1}{a_1} = \frac{m_2}{a_2} \rightarrow m_1 = -0.517, m_2 = -0.276, a_1 = 1.809, a_2 = 0.966 \quad (24)$$

$$y_0 = \frac{N}{a_1 + a_2} \frac{m_1^{m_1} m_2^{m_2}}{(m_1 + m_2)^{m_1+m_2}} \frac{\Gamma(m_1 + m_2 + 2)}{\Gamma(m_1 + 1)\Gamma(m_2 + 1)} = 3.576 \quad (25)$$

$$Mode = Mean - \frac{1}{2} \frac{\mu_2 r + 2}{\mu_2 r - 2} = 3.809 \quad (26)$$

Figure (2) represents sampling distribution for the maximum likelihood estimates.

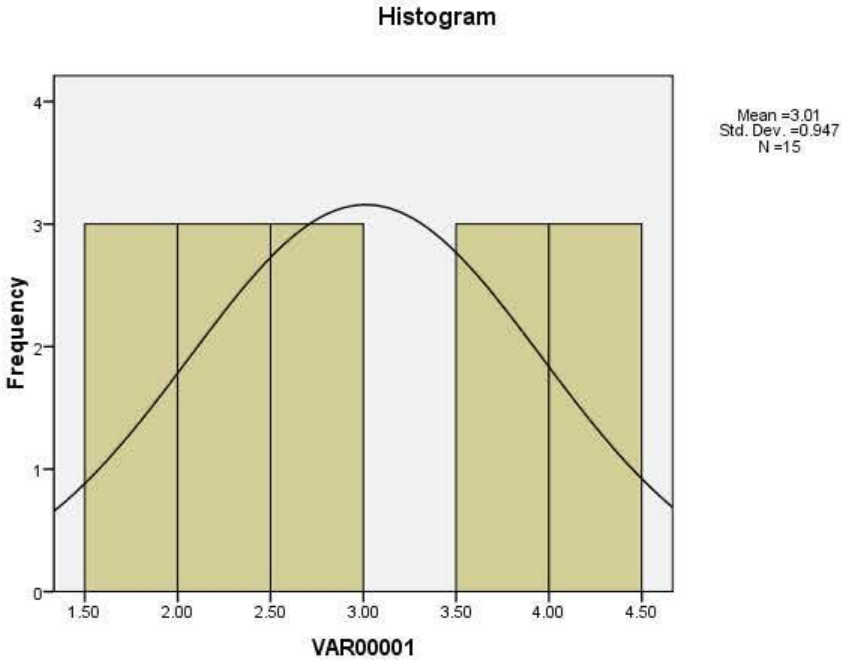

Figure (2) the sampling distribution for the maximum likelihood estimates.

It is concluded that the maximum likelihood estimates are better than the moment estimates so the estimated value is close to the true value.

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