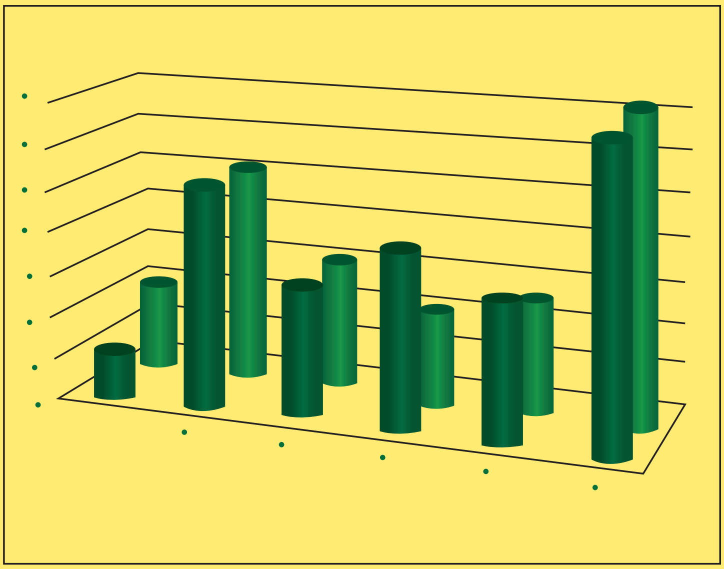
المعهد العربي للتدريب والبحوث الإحصائية

مجلة
العلوم الإحصائية



العدد رقم 30

مجلة علمية محكمة

يصدرها المعهد العربي للتدريب والبحوث الإحصائية

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شروط النشر في مجلة العلوم الإحصائية

1 – تنشر المجلة البحوث والدراسات العلمية في المجالات الإحصائية والمعلوماتية المكتوبة باللغة العربية والانكليزية والفرنسية على أن لا يكون البحث المقدم للنشر قد نشر او قدم للنشر في مجلات او دوريات أخرى او قدم ونشر في دوريات لمؤتمرات أو ندوات.

2 – ترسل البحوث والدراسات الى أمين التحرير على أن تتضمن اسم الباحث او الباحثين وألقابهم العلمية وأماكن عملهم مع ذكر عنوان المراسلة وأرقام الهواتف والبريد الالكتروني. وان يرسل البحث المراد نشره الكترونيا (على قرص او بالبريد الالكتروني) وفق المواصفات أدناه:

أ – أن يكون مطبوعاً على ورق حجم A4 وان يكون على شكل عمود واحد ويستخدم للغة العربية نوع حرف (Simplified Arabic) و(Times New Roman) للإنجليزية والفرنسية وبحجم خط (12). وباستخدام Microsoft Word وعلى وجه واحد للورقة.

ب – الهامش مسافة 2.5 سم لجميع جوانب الورقة.

ج – يرفق الباحث ملخصاً عن بحثه باللغتين العربية والانجليزية والفرنسية بما لا يزيد عن صفحة واحدة.

د - يتم الإشارة الى المصادر العلمية في متن البحث وفي نهايته، مع مراعاة أن لا يتضمن البحث سوى المصادر التي تم الإشارة إليها في المتن ووفق الأصول المعتمدة في ذلك (اسم المؤلف، سنة النشر، عنوان المصدر، دار النشر، البلد).

هـ– ترقم الجداول والرسوم التوضيحية وغيرها حسب ورودها في البحث، كما توثق المستعارة منها بالمصادر الأصلية.

و– أن لا يزيد عدد صفحات البحث اوالدراسة عن (25) صفحة.

3 – يتم إشعار الباحث باستلام بحثه خلال مدة لا تتجاوز يومين عمل من تاريخ استلام البحث.

4 – تخضع كافة البحوث المرسلة الى المجلة للتقييم العلمي الموضوعي ويبلغ الباحث بالتقييم والتعديلات المقترحة إن وجدت خلال مدة لا تتجاوز اسبوعان من تاريخ استلام البحث.

5 – لهيئة تحرير المجلة الحق في قبول او رفض البحث ولها الحق في إجراء أي تعديل او إعادة صياغة جزئية للمواد المقدمة للنشر بما يتماشى والنسق المعتمد في النشر لديها بعد موافقة الباحث.

6 – يصبح البحث المنشور ملكاً للمجلة ولا يجوز إعادة نشره في أماكن أخرى.

7 – تعبر المواد المنشورة بالمجلة عن آراء أصحابها، ولا تعكس وجهة نظر المجلة او المعهد العربي للتدريب والبحوث الإحصائية.

8 – ترسل البحوث على العنوان الالكتروني للمجلة:

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المحتويات

The Topp-Leone Burr II Distribution: Properties and Applications

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Girls Branch Egypt

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The Topp-Leone Burr II Distribution: Properties and Applications

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Abstract

This paper is concerned with introducing a new distribution called Topp-Leone Burr II (TL-BII) distribution. Some statistical properties such as reliability function (rf), hazard rate function (hrf) reversed hazard function (rhrf), the mean, variance, quantile function, median, skewness, and kurtosis are studied. The moments and moment generating function, order statistics are obtained. The method of maximum likelihood is used for estimating the parameters of this distribution.

A simulation study is conducted to represent the flexibility of the distribution and real data sets are used to demonstrate the applicability of the suggested distribution.

Keywords: Topp-Leone, reliability, hazard, skewness, kurtosis.

Introduction

Generating family of distributions is a new improvement for creating and extending the usual classical distributions. Several ways of generating new distributions from classic ones have been developed. Hence, several classes to generate new distributions by adding one or more parameters have been proposed in the statistical literature. For example, Beta-G distribution family that was introduced by Eugene et al. (2002), McDonald class of distributions by Alexander et al. (2012), gamma-G type 1 by Zografos, k. et al. (2009). gamma-G type 2 by Ristic ,M M. and et al. (2012), odd exponentiated generalized by Cordeiro, GM. et al. (2013), exponentiated T-X by Alzaghal A et al. (2013), odd Weibull-G by Bourguignon M. et al. (2014), exponentiated half-logistic by Cordeiro et al. (2014), T-X{Y} quantile-based approach by Aljarrah et al. (2014), Lomax G by Cordeiro et al. (2014), Kumaraswamy-G class of distributions by Cordeiro et al. (2014), Kumaraswamy odd log-logistic-G by Alizadeh et al. (2015), logistic-X by Tahir et al. (2016) and alpha power transformation family of distributions introduced by Mahdavi and Kundu (2017).

These distributions allow us to analyze and interpret the data, identify trends, estimate probabilities, and make informed decisions based on the findings. The choice of a suitable statistical distribution is crucial because it determines how well the model aligns with the observed data and how accurately it represents the real-life behavior of the items. A well-fitting distribution provides valuable information that enables us to draw sound conclusions about the item's performance, predict future outcomes, assess risks, and make informed decisions. In summary, appropriate statistical distributions are essential for modeling the real-life behavior of items because they offer valuable data collected helps us draw meaningful conclusions and make sound decisions based on the analysis of the collected data (Sarker, 2021).

Burr Type II distribution

Burr (1942) suggested twelve forms of cumulative distribution functions which might be useful for fitting data. Abu-Salih, *et al.* (1997) derived Bayes estimators of the parameters of Burr II assuming two priors for namely, quasi-prior and conjugate prior, and considering two types of loss functions.

The random variable X with burr Type II distribution has a *cumulative distribution function* (cdf) given by

$$F(x,\theta) = (1+e^{-x})^{-\theta}, -\infty < X < \infty;\ \theta > 0\ , \qquad (1)$$

The one-parameter burr Type II distribution with *probability density function* (pdf) is defined as:

$$f(x,\theta) = \theta e^{-x}(1+e^{-x})^{-\theta-1}, -\infty < X < \infty;\ \theta > 0\ , \qquad (2)$$

Where θ is the shape parameter.

Topp- Leone Distribution

The Topp-Leone distribution was introduced by Topp and Leone in 1955. The *Topp Leone distribution* (TLD) is a continuous lifetime model having finite support and bathtub (U-shaped) hazard rate function; these features distinguish it from the famous lifetime models such as gamma, Weibull, or Log-normal distribution.

Furthermore, Nadarajah and Kotz (2003) pointed out that the TL distribution provides bathtub shape of the hrf when $0 < \theta < 1$. In addition, if $\theta \geq 1$ the TL distribution has a non-increasing hazard function.

Li (2016) provided the estimation of the parameter of TL distribution under symmetric entropy loss function based on lower record values. Reyad and Othman (2017) studied the Topp–Leone Burr XII distribution properties and applications. EL-Helbawy (2018) studied Bayesian estimation for Topp–Leone Weibull distribution based on dual generalized order statistics and proposed estimation under a finite mixture of two – component Topp–Leone Rayleigh lifetime model.

Behairy *et al.* (2020) studied Topp–Leone inverted Kumaraswamy distribution properties and estimation and prediction.
A random variable$X$is distributed as the TL with parameter α denoted by $X\sim TL(\theta)$, with cdf as

$$F_{TL}(x)=x^{\alpha}\,(2-x)^{\alpha}, \qquad 0<x<1, \quad \alpha>0 \qquad (3)$$

The corresponding (pdf) is

$$f_{TL}(x)=2\alpha\,x^{\alpha-1}(1-x)(2-x)^{\alpha-1}, \qquad 0<x<1, \quad \alpha>0 \quad (4)$$

where θ is a shape parameter
The *reliability function* (rf) , *hazard rate function* (hrf) and *reversed hazard function* (rhrf) are obtained. These functions of the TL distribution are, respectively given by

$$R_{TL}(x)=1-F(x)=1-x^{\alpha}\,(2-x)^{\alpha}, \quad 0<x<1, \quad \alpha>0, \quad (5)$$

$$hf_{TL}(x)=\frac{f(x)}{R(x)}=\frac{2\alpha\,x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{1-x^{\alpha}\,(2-x)^{\alpha}}, \qquad 0<x<1, \quad \alpha>0, \quad (6)$$

and

$$rhf_{TL}(x)=\frac{f(x)}{F(x)}=\frac{2\alpha\,x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{x^{\alpha}\,(2-x)^{\alpha}}=\frac{2\theta\,(1-x)}{x(2-x)}0<x<1, \quad \alpha>0(7)$$

Topp-Leone generalized family

Recently, applying new generators for continuous distributions became more interesting. This methodology can improve fitness and determine tail properties and let the new distribution more flexible to model real data. These features are established by the results of many generators such as beta distribution, generalized Kumaraswamy distribution, generalized beta distribution and the exponentiated family of distributions [See Eugene *et al.* (2002), Jones (2004), Cordeiro and de Castro (2011), Alexander *et al.* (2012), AL-Hussaini and Ahsanullah (2015)].
If a random variable X is distributed as the *(TL)* and bound on [0, 1]. Topp-Leone generalized family of distributions was inferred by Rezaei, *et al.* (2016). Let X be a continuous random variable with (cdf) G(x). The (*TLG*) distribution has (cdf) written by

$$F_{TLG}(x)=\left(1-\left(1-G(x)\right)^{2}\right)^{\alpha} \qquad 0<x<\infty, \ \alpha>0 \qquad (8)$$

By differentiating, the corresponding (pdf) is

$$f_{TLG}(x)=2\alpha\,g(x)(1-G(x))\left(1-\left(1-G(x)\right)^{2}\right)^{\alpha-1}, 0<x<\infty, \ \theta>0 \ (9)$$

where $g(x)=\frac{dG(x)}{dx}$ and where α is a shape parameter.

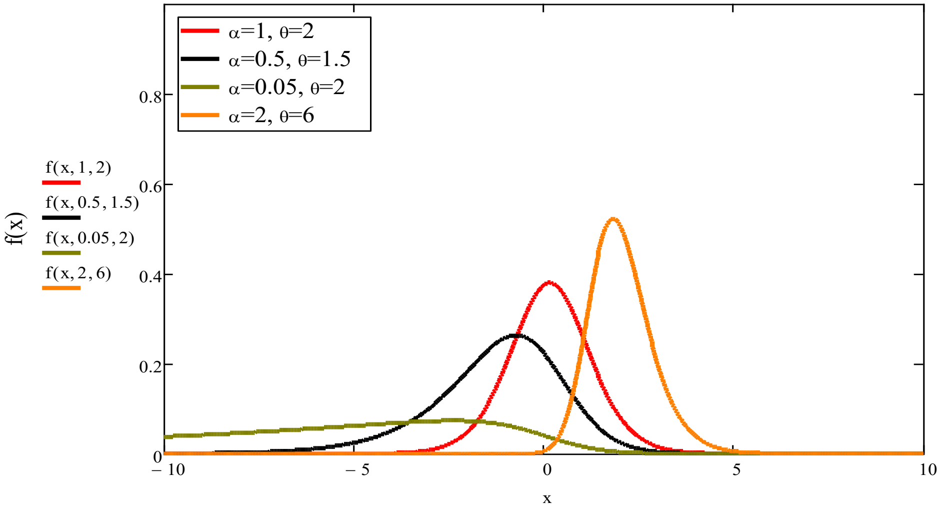
Topp Leone-Burr Type II Distribution

One can construct $\mathrm{Tl}-\mathrm{BII}$ distribution by combining (1) and (2) into (8) and (9 hence the cdf and pdf of the $\mathrm{Tl}-\mathrm{BII}$ are given respectively as follows:

$$F_{(TL-BII)}(x)=\left[1-\left(1-(1+e^{-x})^{-\theta}\right)^{2}\right]^{\alpha}, \quad -\infty<x<\infty, \alpha, \theta>0 \quad (10)$$

$$f(x;\alpha,\theta)=2\alpha\theta e^{-x}(1+e^{-x})^{-\theta-1}\left(1-(1+e^{-x})^{-\theta}\right)\left[1-\left(1-\left((1+e^{-x})\right)^{-\theta}\right)^{2}\right]^{\alpha-1} \quad -\infty<x<\infty, \alpha, \theta>0 \quad (11)$$

Plots of pdf of distribution of different values of the parameters is given in Figure 1.



x

Figure 1: plots of pdf of the $TL-BII$ distribution at different values of the parameters α, θ

From Figure1 it is noticed that the curves are approximately normal increasing and then decreasing.

- The reliability function (rf) of the $TL-BII$ distribution is as follows:

$$R(x;\alpha,\theta)=1-F(x)=1-[1-(1-(1+e^{-x})^{-\theta})^{2}]^{\alpha}, -\infty<X<\infty, \alpha, \theta>0 \quad (12)$$

- The hazard function (hrf) of the $(TL-BII)$ distribution is as follows:

$$h(x;\alpha,\theta)=\frac{f(x;\alpha,\theta)}{R(x;\alpha,\theta)}$$

$$=\frac{2\alpha\theta e^{-x}(1+e^{-x})^{-\theta-1}(1-(1+e^{-x})^{-\theta})\left[1-\left(1-\left((1+e^{-x})\right)^{-\theta}\right)^{2}\right]^{\alpha-1}}{1-[1-(1-(1+e^{-x})^{-\theta})^{2}]^{\alpha}},$$

$$-\infty<X<\infty\,,\alpha,\theta>0 \qquad (13)$$

The plots of h(x) function at different values of the parameters α,θ are displayed in Figure 2.

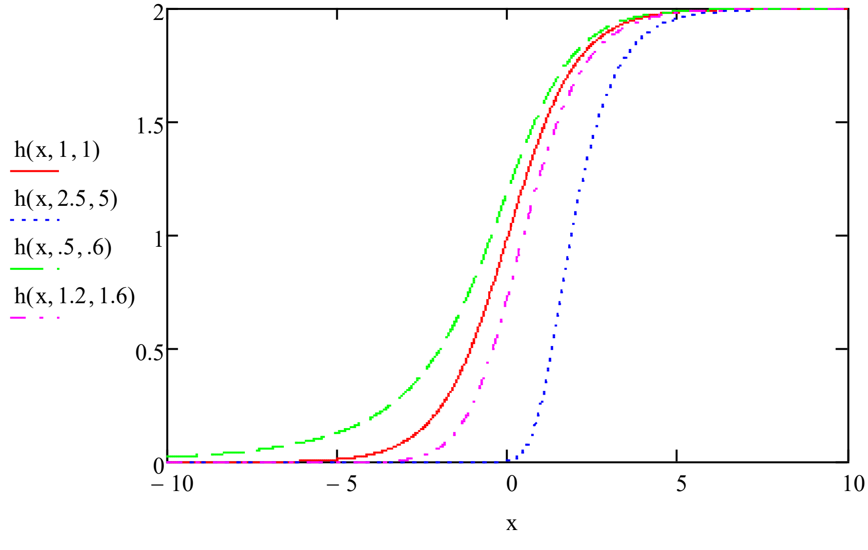


Figure 2: plots of $h(x)$ for the $TL-BII$ distribution at different values of the parameters α,θ

From Figure 2 it is noticed that the curves are increasing at different values for the parameters

- The reversed hazard function rhf is as follows:

$$rh(x;\alpha,\theta)=\frac{f(x;\alpha,\theta)}{F(x;\alpha,\theta)}$$

$$=\frac{2\alpha\theta e^{-x}(1+e^{-x})^{-\theta-1}(1-(1+e^{-x})^{-\theta})\left[1-\left(1-\left((1+e^{-x})\right)^{-\theta}\right)^{2}\right]^{\alpha-1}}{[1-(1-(1+e^{-x})^{-\theta})^{2}]^{\alpha}},$$

$$=2\alpha\theta e^{-x}(1+e^{-x})^{-\theta-1}(1-(1+e^{-x})^{-\theta})\left(1-\left(1-\left((1+e^{-x})\right)^{-\theta}\right)^{2}\right)^{-1}$$

$$-\infty<X<\infty\,,\alpha,\theta>0 \qquad (14)$$

- The cumulative hazard function (Hf) is given by:

$$Hf(x;\alpha,\theta) = -\ln R(x)$$
$$= -\ln\left\{1-\left[1-\left(1-(1+e^{-x})^{-\theta}\right)^2\right]^\alpha\right\}$$
$$= \ln\left\{1-\left[1-\left(1-(1+e^{-x})^{-\theta}\right)^2\right]^\alpha\right\}^{-1}$$
$$= \ln\left[\frac{1}{1-[1-(1-(1+e^{-x})^{-\theta})^2]^\alpha}\right] \qquad (15)$$

- The r^{th} non – central moment of the $TL-BII$ distribution is as follows:

$$\mu_r' = = 2\alpha\theta\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)\sum_{j=0}^{\infty}\binom{2i+1}{j}(-1)^j\sum_{h=0}^{\infty}\binom{-\theta-1-\theta j}{h}\ e^{-xh}\ \frac{2}{(h+1)^{r+1}}\ \Gamma(r+1) \quad (16)$$

$at\ r=1$, the first non – central moment (the mean) is:

$$\mu_1' = 2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}\ e^{-xh}\ \frac{2}{(h+1)^2}$$

$at\ r=2$, the second non – central moment is:

$$\mu_2' = 2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}\ e^{-xh}\ \frac{4}{(h+1)^3} \qquad (18)$$

$at\ r=3$, the third non – central moment is:

$$\mu_3' = 2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}\ e^{-xh}\ \frac{12}{(h+1)^4} \qquad (19)$$

$at\ r=4$, the third non – central moment is:

$$\mu_4' = 2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}\ e^{-xh}\ \frac{48}{(h+1)^5} \qquad (20)$$

- At $r=2$, the variance is as follows:

$$\mu_2 = \mu_2' - {\mu_1'}^2$$
$$=\left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}\ e^{-xh}\ \frac{4}{(h+1)^3}\right) - \left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}\ e^{-xh}\ \frac{2}{(h+1)^2}\right)^2 \qquad (21)$$

- At $r=3$, the third central moment is as follows:

$$\mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2{\mu_1'}^3$$

$$=\left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{12}{(h+1)^4}\right)$$
$$-3\left(\begin{array}{l}\left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{4}{(h+1)^3}\right)\times\\ \left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{2}{(h+1)^2}\right)\end{array}\right)$$
$$+2\left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{2}{(h+1)^2}\right)^3 \tag{22}$$

- At $r =$, the fourth central moment is as follows:

$$\mu_4=\mu_4'-4\mu_3'\mu_1'+6\mu_2'{\mu_1'}^2-3{\mu_1'}^4$$

$$=2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{48}{(h+1)^5}-$$
$$4\left[\begin{array}{l}\left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{12}{(h+1)^4}\right)\times\\ \left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{2}{(h+1)^2}\right)\end{array}\right]+$$
$$6\left[\begin{array}{l}\left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{4}{(h+1)^3}\right)\times\\ \left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{2}{(h+1)^2}\right)^2\end{array}\right]-$$
$$3\left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{2}{(h+1)^2}\right)^4$$

α_3 is obtained as follows:

$$\alpha_3=\frac{\mu_3}{\mu_2^{3/2}}=\left(\boldsymbol{2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{12}{(h+1)^4}}\right)\mathbf{3}$$
$$\left(\begin{array}{l}\left(\boldsymbol{2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{4}{(h+1)^3}}\right)\\ \left(\boldsymbol{2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{2}{(h+1)^2}}\right)\end{array}\right)+$$
$$\mathbf{2}\left(\boldsymbol{2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{2}{(h+1)^2}}\right)^{\mathbf{3}}/$$
$$\left\{\left(\boldsymbol{2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{4}{(h+1)^3}}\right)-\right.$$
$$\left.\left(\boldsymbol{2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{2}{(h+1)^2}}\right)^{2}\right\}^{\frac{3}{2}}$$

- The coefficient of kurtosis

α_4 is obtained as follows:

$$\alpha_4=\frac{\mu_4}{\mu_2^2}2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{48}{(h+1)^5}-$$
$$4\left[\begin{array}{l}\left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{12}{(h+1)^4}\right)\times\\ \left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}e^{-xh}\frac{2}{(h+1)^2}\right)\end{array}\right]+$$

$$6\left[\begin{array}{l}\left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}\ e^{-xh}\ \frac{4}{(h+1)^3}\right)\times\\ \left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}\ e^{-xh}\ \frac{2}{(h+1)^2}\right)^2\end{array}\right]-$$
$$3\left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}\ e^{-xh}\ \frac{2}{(h+1)^2}\right)^4 /$$
$$\left(\left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}\ e^{-xh}\ \frac{4}{(h+1)^3}\right)-\right.$$
$$\left.\left(2\alpha\theta\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{2i+1}{j}(-1)^{i+j}\binom{-\theta-1-\theta j}{h}\ e^{-xh}\ \frac{2}{(h+1)^2}\right)^2\right)^2$$

- The quantile function of the $TL-BII$ distribution is obtained as follows:

$$F(x,\alpha,\theta)=q$$

$$\left(1-\left[1-(1+e^{-x})^{-\theta}\right]^2\right)^{\alpha}=q\ \therefore x_q=-\ln\left[\left(1-\left(1-q^{1/\alpha}\right)^{1/2}\right)^{-1/\theta}-1\right]$$

At $=0.5$, the median of the $(TL-BII)$ distribution is given by:

$$m=-\ln\left[\left(1-\left(1-0.5^{1/\alpha}\right)^{1/2}\right)^{-1/\theta}-1\right]\qquad(26)$$

- The mode of the $(TL-BII)$ distribution is obtained using the following equation:

$$\begin{aligned}f'(x)=0\to f'(x)&\\ &=2\alpha\theta\Big[e^{-x}(1+e^{-x})^{-\theta\alpha-1}\left[2-(1+e^{-x})^{-\theta}\right]^{\alpha-1}\theta(1+e^{-x})^{-\theta-1}\\ &+e^{-x}(1+e^{-x})^{-\theta\alpha-1}\left(1-(1+e^{-x})^{-\theta}\right)(\alpha\\ &-1)\left[2-(1+e^{-x})^{-\theta}\right]^{\alpha-2}\theta(1+e^{-x})^{-\theta-1}(-e^{-x})\\ &+e^{-x}\left[2-(1+e^{-x})^{-\theta}\right]^{\alpha-1}\left(1-(1+e^{-x})^{-\theta}\right)(-\theta\alpha\\ &-1)(1+e^{-x})^{-\theta\alpha-2}(-e^{-x})\\ &+(1+e^{-x})^{-\theta\alpha-1}\left[2-(1+e^{-x})^{-\theta}\right]^{\alpha-1}\left(1-(1+e^{-x})^{-\theta}\right)(-e^{-x})\Big]\\ &=0\qquad(27)\end{aligned}$$

It is noticed that the mode cannot be obtained in closed form. So that it can be obtained numerically using R program.
The results of the mode are displayed in Table 1.

Table1.: some values of the mode at different values of the parameters α and θ

α	θ	Mode
10	7	3.064843
4	6	2.45739
1	1	-0.6931469
0.5	1	-1.270219

- **Order statistics**

Let $X_1, X_2, \dots, X_n$ be $\mathrm{i.i.d.}$ random variables from the $TL - BII\ (\alpha, \theta)$ distribution and X_i denote the i^{th} order statistic. Then the pdf of X_i from $TL - BII\ (\alpha, \theta)$ is

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x), -\infty < x_{(1)} < x_{(2)} < \cdots < \infty$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\left[1-\left(1-(1+e^{-x})^{-\theta}\right)^2\right]^{\alpha}\right]^{r-1}\left[1 - \left[1-\left(1-(1+e^{-x})^{-\theta}\right)^2\right]^{\alpha}\right]^{n-r} 2\alpha\theta e^{-x}(1+e^{-x})^{-\theta-1}\left(1-(1+e^{-x})^{-\theta}\right)\left[1-\left(1-\left((1+e^{-x})\right)^{-\theta}\right)^2\right]^{\alpha-1} \quad (28)$$

- At $r = 1$ the smallest value of the order statistics is

$$f_{1:n}(x) = n[1-F(x)]^{n-1} f(x), -\infty < x_{(1)} < x_{(2)} < \cdots < \infty$$

$$= n\left[1-\left[1-\left(1-(1+e^{-x})^{-\theta}\right)^2\right]^{\alpha}\right]^{n-1} 2\alpha\theta e^{-x}(1+e^{-x})^{-\theta-1}\left(1-(1+e^{-x})^{-\theta}\right)\left[1-\left(1-\left((1+e^{-x})\right)^{-\theta}\right)^2\right]^{\alpha-1}, -\infty < x_{(1)} < \infty \quad (29)$$

- At $r = n$ the largest value of the order statistics is

$$f_{n:n}(x) = n[1-F(x)]^{n-1} f(x), -\infty < x_{(1)} < x_{(2)} < \cdots < \infty$$

$$= n\left[\left[1-\left(1-(1+e^{-x})^{-\theta}\right)^2\right]^{\alpha}\right]^{r-1} 2\alpha\theta e^{-x}(1+e^{-x})^{-\theta-1}\left(1-(1+e^{-x})^{-\theta}\right)\left[1-\left(1-\left((1+e^{-x})\right)^{-\theta}\right)^2\right]^{\alpha-1}, -\infty \le x_{(n)} \le \infty \quad (30)$$

- **Maximum likelihood estimation:**

In this subsection, the ML method is used to estimate the shape parameters , θ , rf and hrf based on Type II censored sample of $TL - BII$ (α, θ) distribution. The confidence intervals (CIs) of the shape parameters, rf and hrf are obtained.

Suppose that $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ is a Type II censored sample of size r obtained from a life-test on n items whose lifetimes have a $TL - BII$ (α, θ) distribution. The LF is given by

- $L(\alpha, \theta; x) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f\left(x_{(i)}, \alpha, \theta\right)\left[R(x_{(r)}; \alpha, \theta\right]^{n-r},$

$$= \frac{n!}{(n-r)!}\prod_{i=1}^{r} 2\alpha\theta e^{-x_{(i)}}(1+e^{-x_{(i)}})^{-\theta-1}\left(1-(1+e^{-x_{(i)}})^{-\theta}\right)\left[\left[1-\left(1-(1+e^{-x_{(r)}})^{-\theta}\right)^2\right]^{\alpha}\right]^{n-r}. \quad (31)$$

the natural logarithm of $L(\alpha, \theta; x)$ is given by

$$l \propto r\ln(2\alpha\theta) - \sum x_i - (\theta+1)\sum_{i=1}^{r}\ln(1+e^{-x_{(i)}}) - \sum_{i=1}^{r}\ln\left(1-(1+e^{-x})^{-\theta}\right) + (n-r)\ln[1-(1-(1+e^{-x_{(r)}})^{-\theta})^2]^{\alpha} \qquad (32)$$

Equations 31,32 are differentiating with respect to the parameters α, θ and equating zero.

Since these equations cannot be solved in closed form, so that the solutions are obtained numerically using program Mathematica.

Renyi Entropy

Entropy is a measure of randomness of a random variable, since the entropy of variable X is the amount of information on uncertainty contained in a random observation regarding its distribution. A large value of entropy indicates that greater uncertainty in the data.

For a random variable X with TL-BII distribution the Renyi entropy is obtained as follows:

$$R_\delta(f) = \frac{2}{1-\delta}(2\alpha\theta)^{\delta}\sum_{i,j,h=0}^{\infty}\binom{\alpha-1}{i}\binom{\delta+2i\delta}{j}\binom{-\theta\delta-\delta-\theta j}{h}(-1)^{i+j}\frac{1}{(\delta+h)} \qquad (33)$$

Numerical study:

Simulation Study

A simulation study is conducted to evaluate the performance of ML estimators of parameters, rf and hrf for TL-BII distribution using Mathematica 12 by the following steps:

1. Generating random samples of sizes $(n = 30, 50, 70.100.150{,}200)$ based on Type II censored sample {0% (complete sample) and 90% at $(\theta = 1.1\,, \alpha = 1.2)$.
2. The transformation between the uniform distribution and TL-BII distribution is
$$x = -\ln\left[\left(1-\left(1-u^{1/\alpha}\right)^{1/2}\right)^{-1/\theta} - 1\right]$$
3. Repeat the previous steps m times where $m = 1000$ is the number of repetitions
4. The performance of the resulting estimators is considered using MSE, RMSE and the length of the confidence interval.

The result of the simulation is displayed in Tables 1 and 2.

Table 1 The ML estimates, MSE, RMSE variance, bias and 95% confidence intervals of the parameters of the TL-BII distribution for different sample sizes in the case of complete sample (0% censoring)

n	r	parameter	MLE	MSE	RMSE	Variance	Bias	CI		
								UL	LL	Length
30	30	θ	**1.10728**	**0.028877**	**0.154486**	**0.0288249**	**0.00727828**	**1.44005**	**0.774511**	**0.665534**
		α	**1.19319**	**0.0145723**	**0.100597**	**0.014526**	**-0.00680533**	**1.42942**	**0.956968**	**0.472454**
50	50	θ	**1.03116**	**0.022073**	**0.135063**	**0.0173334**	**-0.0688441**	**1.2892**	**0.773109**	**0.516094**
		α	**1.23071**	**0.0080214**	**0.074635**	**0.0070785**	**0.0307066**	**1.39561**	**1.0658**	**0.329807**
70	70	θ	**1.04428**	**0.0122251**	**0.100516**	**.00912008**	**-0.0557227**	**1.23146**	**0.857099**	**0.374357**
		α	**1.16964**	**0.0072451**	**0.070932**	**0.0063234**	**-0.0303599**	**1.3255**	**1.01378**	**0.311719**
100	100	θ	**1.03156**	**0.0159734**	**0.114896**	**0.0112897**	**-0.068437**	**1.23982**	**0.823307**	**0.416513**
		α	**1.20295**	**0.0068324**	**0.068882**	**0.0068237**	**0.00295272**	**1.36486**	**1.04105**	**0.323814**
150	150	θ	**1.08044**	**0.0094352**	**0.088304**	**0.0090525**	**-0.0195621**	**1.26692**	**0.893954**	**0.372968**
		α	**1.18582**	**0.0068390**	**0.068915**	**0.0066379**	**-0.0141807**	**1.34551**	**1.02613**	**0.319379**
200	200	θ	**1.11297**	**0.015623**	**0.113627**	**0.0154542**	**0.0129673**	**1.35662**	**0.86931**	**0.487314**
		α	**1.17978**	**0.0063597**	**0.066456**	**0.0059508**	**-0.0202218**	**1.33098**	**1.02858**	**0.302396**

Table 2 The ML estimates, MSE, variance, bias and 95% confidence intervals of the parameters of the TL-BII distribution for different sample sizes in the case of complete sample (90% censoring), ($\theta = 1.1$, $\alpha = 1.2$)

n	r	parameter	MLE	MSE	RMSE	Variance	Bias	CI		
								UL	LL	Length
30	**27**	$\boldsymbol{\theta}$	**1.10728**	**0.288779**	**0.154486**	**0.0288249**	**0.0072782**	**1.44005**	**0.774511**	**0.665534**
		$\boldsymbol{\alpha}$	**1.19319**	**0.0145723**	**0.100597**	**0.014526**	**-0.0068053**	**1.42942**	**0.956968**	**0.472454**
50	45	θ	**1.08799**	**0.0258783**	**0.146243**	**0.0257341**	**-0.0120066**	**1.40241**	**0.773573**	**0.628841**
		α	**1.20462**	**0.0089391**	**0.078789**	**0.0089177**	**0.004624**	**1.38971**	**1.01953**	**0.37018**
70	63	θ	**1.09171**	**0.0170364**	**0.118658**	**.0169678**	**-0.0082868**	**1.34702**	**0.836403**	**0. 5106**
		α	**1.19212**	**0.0078600**	**0.0738808**	**0.0077979**	**-0.0078799**	**1.3652**	**1.01904**	**0.3461**
100	90	θ	**1.05417**	**0.0143508**	**0.108904**	**0.01225**	**-0.045834**	**1.2711**	**0.837234**	**0.433865**
		α	**1.21239**	**0.0045463**	**0.0561891**	**0.0043927**	**0.012394**	**1.3423**	**1.08249**	**0.25981**
150	135	θ	**1.01395**	**0.0229385**	**0.137686**	**0.0155342**	**-0.0860486**	**1.25824**	**0.769664**	**0.48857**
		α	**1.21343**	**0.0072685**	**0.0710465**	**0.0070882**	**0. 134261**	**1.37844**	**1.04841**	**0.33003**
200	180	θ	**1.07079**	**0.0089407**	**0.0859596**	**0.0080876**	**-0.0292078**	**1.24706**	**0.894527**	**0.352531**
		α	**1.20021**	**0.0047183**	**0.0572416**	**0.0047182**	**0.0002122**	**1.33484**	**1.06558**	**0.269263**

From Tables 3,4 it is observed that:

- As the sample size n increases, the ML estimates of the TL-BII distribution become close to the population parameter values which are ($\theta = 1.1$, $\alpha = 1.2$), in most cases.
- The MSE, RMSE variance and length of the CIs decrease, the bias approaches to zero and CIs become narrower, in most cases, as the sample size increases.

An application using real data

This application is given by Dulal K. Bhaumlk *et al*. (2009). The data refers to the vinyl chloride data obtained from clean upgradient monitoring wells in mg/l:

Real data set is displayed as follows

5.1	1.2	1.3	0.6	0.5	2.4	0.5
1.1	8	0.4	2	0.5	5.3	3.2
2.7	2.9	2.5	2.3	1.8	0.9	2
4	6.8	1.2	0.4	0.2	0.8	0.6
0.1	0.9	0.1	1	0.4	0.2	

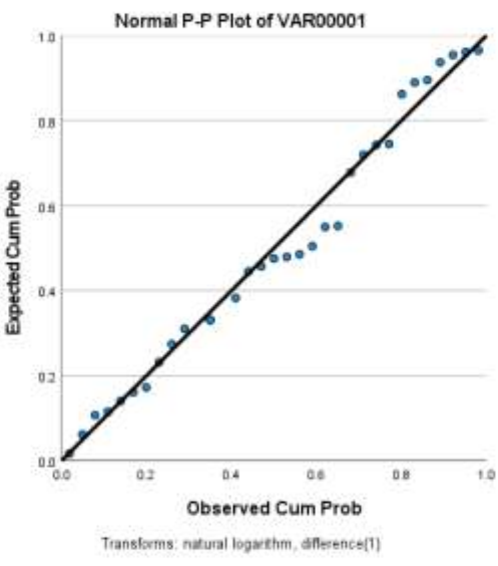
The descriptive statistics of this data are obtained in Table 3

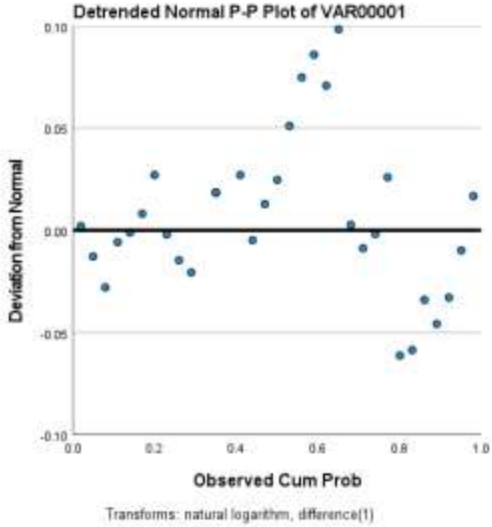
Table 3: the descriptive statistics

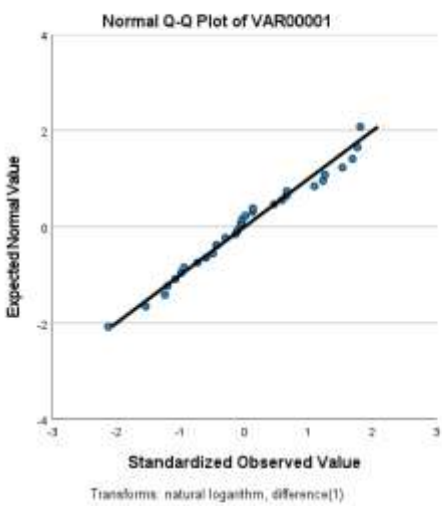
n	Min	Max	Range	Mean	Standard deviation	mode	Skewness	Kurtosis
34	1	8	7.9	1.8794	1.95259	0.4	1.679	2.535

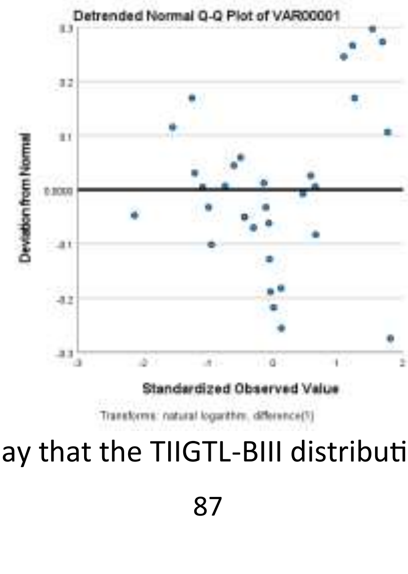
From table 3 it is noticed that the skewness measure is positive, which indicates that the distribution of the data is skewed to the right. Also, the kurtosis measure value is less than 3, thus the distribution of the data indicates a platykurtic distribution, meaning the data spread out with a flatter peak.

Plots of P.P. and Q.Q. are displayed in Figure 3,4









Figures 3 and 4 display that the TIIGTL-BIII distribution fits the data.

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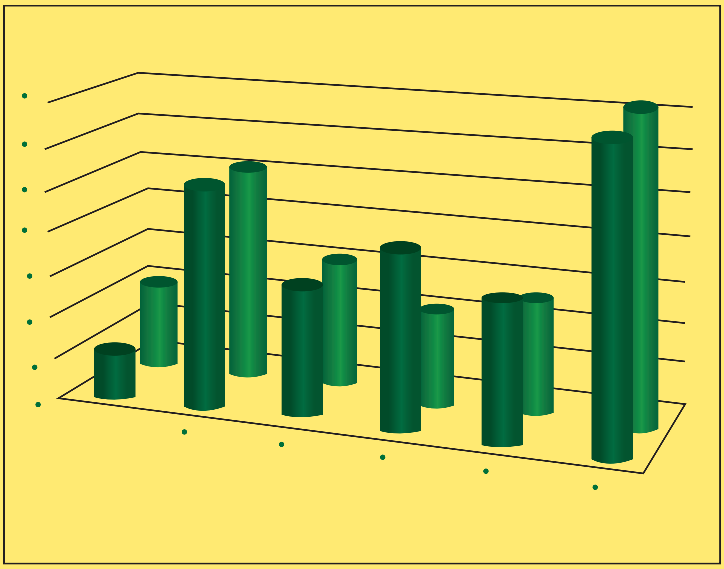

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