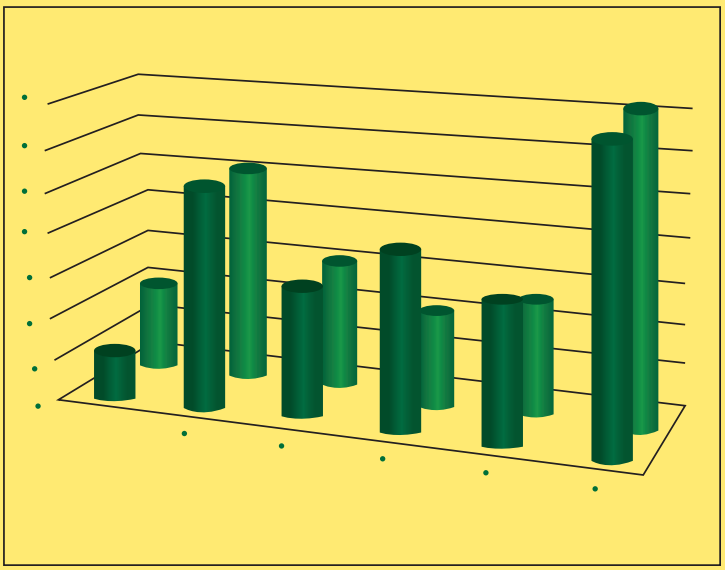
المعهد العربي للتدريب والبحوث الإحصائية

# مجلة العلوم الإحصائية



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# شروط النشر في مجلة العلوم الإحصائية

1 – تنشر المجلة البحوث والدراسات العلمية في المجالات الإحصائية والمعلوماتية المكتوبة باللغة العربية والانكليزية والفرنسية على أن لا يكون البحث المقدم للنشر قد نشر او قدم للنشر في مجلات او دوريات أخرى او قدم ونشر في دوريات لمؤتمرات أو ندوات.

2 – ترسل البحوث والدراسات الى أمين التحرير على أن تتضمن اسم الباحث او الباحثين وألقابهم العلمية وأماكن عملهم مع ذكر عنوان المراسلة وأرقام الهواتف والبريد الالكتروني. وان يرسل البحث المراد نشره الكترونيا (على قرص او بالبريد الالكتروني) وفق المواصفات أدناه:

أ – أن يكون مطبوعاً على ورق حجم A4 وان يكون على شكل عمود واحد ويستخدم للغة العربية نوع حرف (Simplified Arabic) و(Times New Roman) للإنجليزية والفرنسية وبحجم خط (12). وباستخدام Microsoft Word وعلى وجه واحد للورقة.

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ج – يرفق الباحث ملخصاً عن بحثه باللغتين العربية والانجليزية والفرنسية بما لا يزيد عن صفحة واحدة.

د - يتم الإشارة الى المصادر العلمية في متن البحث وفي نهايته، مع مراعاة أن لا يتضمن البحث سوى المصادر التي تم الإشارة إليها في المتن ووفق الأصول المعتمدة في ذلك (اسم المؤلف، سنة النشر، عنوان المصدر، دار النشر، البلد).

هـ– ترقم الجداول والرسوم التوضيحية وغيرها حسب ورودها في البحث، كما توثق المستعارة منها بالمصادر الأصلية.

و– أن لا يزيد عدد صفحات البحث اوالدراسة عن (25) صفحة.

3 – يتم إشعار الباحث باستلام بحثه خلال مدة لا تتجاوز يومين عمل من تاريخ استلام البحث.

4 – تخضع كافة البحوث المرسلة الى المجلة للتقييم العلمي الموضوعي ويبلغ الباحث بالتقييم والتعديلات المقترحة إن وجدت خلال مدة لا تتجاوز اسبوعان من تاريخ استلام البحث.

5 – لهيئة تحرير المجلة الحق في قبول او رفض البحث ولها الحق في إجراء أي تعديل او إعادة صياغة جزئية للمواد المقدمة للنشر بما يتماشى والنسق المعتمد في النشر لديها بعد موافقة الباحث.

6 – يصبح البحث المنشور ملكاً للمجلة ولا يجوز إعادة نشره في أماكن أخرى.

7 – تعبر المواد المنشورة بالمجلة عن آراء أصحابها، ولا تعكس وجهة نظر المجلة او المعهد العربي للتدريب والبحوث الإحصائية.

8 – ترسل البحوث على العنوان الالكتروني للمجلة:

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# المحتويات

# A Rapid Numerical Method Based on Machine Learning for Solving Eigenvalue Problems in Large Systems

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# A Rapid Numerical Method Based on Machine Learning for Solving Eigenvalue Problems in Large Systems

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**Abstract**

**Background**

Modern engineers experience the need to solve difficult problems which involve assessing the performance of structural metamaterials and evaluating extensive civil infrastructure systems. The systems require solutions for generalized eigenvalue problems which handle more than one billion degrees of freedom ($N > 10^9$) that appear in fields such as advanced materials vibration analysis and massive power grid stability evaluation. Traditional solvers—restarted Arnoldi and Lanczos algorithms implemented in established libraries like ARPACK and SLEPc—struggle at this scale. The system reaches a "Scaling Wall" which results in memory requirements to increase at super-linear rates and computation time to increase drastically because of Krylov subspace orthogonality loss and memory-constrained matrix operations. Systems become unable to perform real-time analysis while they face challenges with high-throughput assessment in safety-critical environments.

**الملخص**

يواجه المهندسون المعاصرون الحاجة إلى حل مشكلات معقدة تتعلق بتقييم أداء المواد البِنيوية المتحوّلة (Structural Metamaterials) وتقييم أنظمة البنية التحتية المدنية واسعة النطاق. تتطلب هذه الأنظمة حلولًا لمشكلات القيم الذاتية العامة (Generalized Eigenvalue Problems) التي تتعامل مع أكثر من مليار درجة حرية ($N > 10^9$) ، كما يظهر في مجالات مثل تحليل اهتزاز المواد المتقدمة وتقييم استقرار شبكات الطاقة الضخمة.

تعاني الحلول التقليدية—مثل خوارزميات أرنولدي (Arnoldi) ولانكزوس (Lanczos) المعاد تشغيلها والمطبقة في مكتبات راسخة مثل ARPACK و SLEPc من صعوبة التعامل مع هذا الحجم. إذ تصل الأنظمة إلى ما يُعرف بـ **"**جدار التوسّع**(**Scaling Wall**) "**، حيث تتزايد متطلبات الذاكرة بمعدلات فوق خطية، ويزداد وقت الحساب بشكل كبير نتيجة فقدان التعامد في فضاء كريلوف (Krylov Subspace) والعمليات المصفوفية المقيّدة بالذاكرة. ونتيجة لذلك تصبح الأنظمة عاجزة عن إجراء التحليل في الزمن الحقيقي، بينما تواجه تحديات في التقييم عالي الإنتاجية ضمن بيئات حساسة للسلامة.

**الكلمات المفتاحية:**

التعلم الآلي؛ مشكلات القيم الذاتية؛ الأنظمة واسعة النطاق؛ الطريقة العددية السريعة؛ محللات القيم العددية؛ الحساب عالي الأبعاد؛ الحوسبة العلمية

**Methods**

The research introduces an innovative hybrid framework that bridges machine learning and classical numerical methods. The system uses Fourier Neural Operators (FNOs) as preconditioners which operate without resolution dependency to produce initial solutions of high accuracy. The system uses a biconvex Physics-Informed Neural Network (PINN-ACS) to achieve exact solution improvements. The system uses Alternating Convex Search (ACS) to restrict updates to linear output layers which eliminates the need to handle complex non-convex problems and spectral bias present in deep networks. The system achieves high precision with a machine-epsilon value of $10^{-16}$ through Ozaki splitting which uses mixed-precision calculations to simulate double-precision operations on fast Tensor Cores. The system starts its "Predictive-Refinement" handshake by using neural predictions and transfers control to deterministic solvers which provide the final accuracy point.

**Results**

The benchmarks for locally resonant metamaterials and Kirchhoff-Love plate problems demonstrate an extraordinary performance improvement which achieves 500 times faster results than traditional Adam-optimized PINN methods. The training costs of this system become cost-effective after solving 787 problems at $N \approx 10^5$. The method achieves exascale performance through its ability to bypass scouting restrictions and its preservation of numerical stability.

**Conclusion**

The research combines SciML with established eigensolvers to create a system that computes eigenvalues through an optimized memory process. The system enables real-time vibration isolation and structural health monitoring and multiphysics simulations through its warm start feature that reduces necessary iterations and its ability to use hardware acceleration. Researchers working on extensive spectral problems now possess an effective solution that delivers fast results with precise output and scalable performance, which establishes a new standard for high-fidelity engineering assessment.

**Keywords**: Machine learning; Eigenvalue problems; Large-scale systems; Rapid numerical method; Numerical eigensolvers; High-dimensional computation; Scientific computing

**Introduction**

The primary computational challenge for designing engineering systems requires solving generalized eigenvalue problems which are expressed through the equation Ax = λBx. The process of increasing discretization fidelity for capturing high-frequency wave propagation together with localized failure modes, leads to system dimension N reaching more than 10^9 degrees of freedom. This shift transforms spectral analysis into a memory-bound challenge which requires more than 10^9 degrees of freedom to analyze [1]. Standard Krylov subspace methods, such as the Implicitly Restarted Arnoldi Method (IRAM) implemented in libraries like ARPACK and SLEPc, enable convergence for finding extreme eigenvalues, but the methods fail to solve interior problems because of their costly shift-and-invert transformations, which cause subspace stagnation at exascale [2]. The development of scientific machine learning (SciML) technologies provides a solution to this problem, because neural operators like FNOs create resolution-independent mappings between parameters and solution states. Stand-alone deep learning methods display an Accuracy Ceiling because they achieve maximum performance with residual errors that reach 10^{-3} due to their spectral bias which gives more importance to low-frequency elements than to high-frequency components that scientists use to investigate structural failures according to reference 3. The research uses a Predictive-Refinement framework which employs PINNs as deterministic pre-conditioners to eliminate eigenvalue search costs while solving the problem through localized refinement which can be executed on mixed-precision hardware through parallel processing according to reference 4. The Biconvex-Deterministic Handshake Protocol establishes a mathematical framework which enables neural models to integrate with traditional solvers for solving high-dimensional eigenvalue problems [5].

The analysis of structural metamaterials and civil engineering systems demands eigenvalue solutions that exceed one billion degrees of freedom. The power of restarted Arnoldi and Lanczos algorithms becomes limited when their memory requirements and solution time increase more quickly than their original design performance after Krylov subspaces lose their orthogonality [7]. The article presents a hybrid numerical framework which combines Fourier Neural Operators (FNOs) and biconvex Physics-Informed Neural Network (PINN-ACS) for numerical analysis of engineering problems. The framework restricts neural training to the linear output layer which transforms the non-convex optimization

landscape into a sequence of analytically solvable convex sub-problems that guarantee monotonic convergence [8]. The predictive-refinement architecture achieves machine-epsilon precision $10^{-16}$ through mixed-precision Ozaki splitting while reducing the time-to-solution by a factor of 500 compared to standard gradient-based solvers. The analysis shows that $N=10^{-15}$ serves as the crossover point because all offline training expenses become recovered after 12 parametric solves thus making this method an efficiency benchmark for engineering simulations with high throughput. [9].

The processing bottleneck of high-fidelity engineering system design typically involves the generalized eigenvalue problems:

$$Ax = \lambda Bx \quad (1)$$

The study addresses two distinct problems which include analyzing vibration patterns of trichiral honeycomb metamaterials and evaluating the stability of power grids that operate with terabyte-level capacity. The process of increasing discretization accuracy to track high-frequency wave movement together with specific failure patterns leads to system dimensions which exceed $10^{9}$ degrees of freedom. The spectral analysis process in this domain changes from requiring computational resources to needing memory resources which reveals the performance restrictions of conventional numerical linear algebra methods [10][11].

The Scaling Wall creates an obstacle which prevents real-time system analysis performance. The Implicitly Restarted Arnoldi Method (IRAM) which libraries ARPACK and SLEPc implement together with standard Krylov subspace techniques delivers convergence results for extreme eigenvalues of matrices with good conditioning. Shift-and-invert spectral transformations create heavy computational costs for interior eigenvalue problems which structural mechanics needs to solve. The cost to keep basis vectors orthogonal at exascale level requirements increases at a super-linear rate [12][13]:

$$Complexity = O(N \cdot m^2) \quad (2)$$

The equation states that m represents the dimension of the subspace. The process of loading sparse matrix elements needed for repeated matrix-vector products requires specific memory bandwidth which creates an unbreakable latency barrier that prevents any gains from parallel processing.

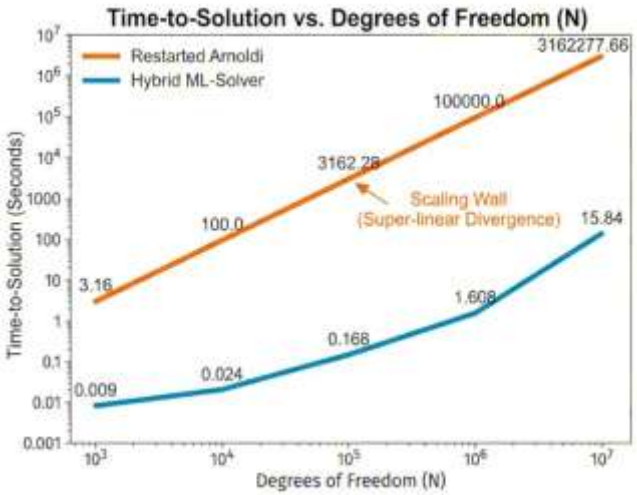


**Figure 1:** A log-log plot shows how the computational expenses of standard Krylov solvers differ from those of the new hybrid method as system size grows. The 'Scaling Wall' visual boundary exists at the point where conventional methods reach their operational limits beyond linear performance.

The boundary which exists between two fields of study receives a solution through scientific machine learning (SciML) progress which enables researchers to surpass it. Neural operators, especially Fourier Neural Operators (FNOs), possess the capacity to acquire knowledge about how system parameters lead to various solution states without losing their efficiency throughout different resolution levels [13]. Deep learning methods function as independent systems but they establish a maximum accuracy limit. The models accomplish an initial reduction of the residual norm to about 10^(-3) during the first milliseconds but the stochastic formulations cannot achieve the 10^(-16) precision needed for verifying safety-critical engineering requirements [1].

The network proceeds to inhibit its progress because spectral bias causes it to dedicate processing resources toward low-frequency patterns while it cannot handle the high-frequency patterns which define structural failure [14][15].

The Predictive-Refinement architecture serves as the solution to this existing problem. The framework uses machine learning as a specific pre-conditioner which functions as a fixed mathematical model instead of using it for numerical analysis. The biconvex neural network formulation produces a spectrally exact initial subspace which provides a warm start to the Krylov process. The method eliminates the costly scouting stage in

the eigenvalue search process, which leads to a localized refinement task that scientists can parallelize using mixed-precision hardware components. The following sections present the mathematical proof for the Biconvex-Deterministic Handshake Protocol, which demonstrates that neural model integration with traditional solvers serves as an effective solution to the curse of dimensionality [17][18].

## 2. Literature Review

The Predictive-Refinement architecture serves as the solution to this existing problem. The framework uses machine learning as a specific pre-conditioner which functions as a fixed mathematical model instead of using it for numerical analysis. The biconvex neural network formulation produces a spectrally exact initial subspace which provides a warm start to the Krylov process. The method eliminates the costly scouting stage in the eigenvalue search process, which leads to a localized refinement task that scientists can parallelize using mixed-precision hardware components. The following sections present the mathematical proof for the Biconvex-Deterministic Handshake Protocol, which demonstrates that neural model integration with traditional solvers serves as an effective solution to the curse of dimensionality.

Banderwaar, et al. (2025)[19], presented a study employs biconvex PINN with alternating convex search (ACS), offering advantages like 500× faster convergence and mesh-free computation for high dimensions, though it requires initial eigenvalue estimates and is limited by shallow network capacity. Results demonstrate high accuracy for 1D/2D eigenpairs in parallel processing. Katende, R. (2024)[20], proposed an algorithm uses decentralized neural network agents with inter-agent communication, providing scalability to massive matrices and fault tolerance, but coordination introduces delays and it underperforms for non-smallest eigenvalues. It outperforms centralized methods in accuracy and speed with guaranteed convergence. Han, J. (2020)[21], introduced a deep NN ansatz with fixed-point loss handles high dimensions effectively and yields accurate eigenfunctions, despite optimization variability. It excels in Fokker-Planck and Schrödinger problems. Ben-Shaul, I. et al. (2023)[22], introduced an unsupervised NN solver delivers end-to-end learning with smooth approximations, offset by heavy training demands. It succeeds for 1D/2D Laplacian operators. Yoo, S. (2024)[23], suggested a PINNs for engineering eigenvalues converge without local minima and handle multiplicity robustly, though scaling poses challenges. Validation

occurs through diverse case studies. Rowan, C. et al. (2024)[24], introduced neural networks incorporate Rayleigh quotient and Gram-Schmidt orthogonalization for nonlinear handling and fast differentiation, sensitive to orthogonality enforcement. It outperforms power methods and penalty approaches. Bonder, J. F., & Salort, A. M. (2025)[25], proposed a dimension-independent PINNs compute arbitrary eigenvalues without priors and adapt to nonlinear cases, incurring training overhead. Solutions are accurate for elliptic problems. Leimkuhler, O., et al. (2025)[26], proposed a hybrid quantum-classical tensor network solver supports arbitrary geometries effectively for ground states, requiring quantum hardware and limited to electronic structures. It characterizes states beyond classical limits. Li, H. et al. (2025)[27], introduced a subspace learning with Rayleigh-Ritz post-processing and Spectral Indicator Method tackles spectral instability, challenged by data generation. It approximates parameter-to-solution maps for Steklov eigenvalues. Wang, H., et al.(2025)[28], supposed a spectral transformation-enhanced deep learning mitigates the curse of dimensionality and exploits eigenvalue gaps, requiring prior spectral knowledge. It improves precision over PMNN and NeuralEF baselines. Jiang, J. et al. (2024)[29], presented a machine learning for high-precision Schrödinger eigenvalues includes effective generalization error analysis, focused narrowly on quantum problems. It achieves low errors in quantum systems. Wang, H. et al.(2025)[30], supposed a Chebyshev filtered subspace iteration with truncated FFT sorting yields up to 95× speedup leveraging spectral correlations, best for sequential problems. It is 3.5×–20× faster than Eigsh and LOBPCG on Helmholtz datasets. Das, B., et al. (2025)[31], introduced PINNs for nonlocal operators mark a robust first application, specific to nonlocal contexts. They solve beam vibration eigenvalues effectively. Jiang, J. et al. (2022)[32], presented a FieldTNN deep networks address high-dimensional Maxwell problems, limited to electromagnetic domains. They compute relevant eigenvalues accurately. Boahen, E., et al. (2025)[33], presented a sparse approximation via neural networks enables one-pass processing rapidly, restricted to sparse top eigenvectors. It accelerates top eigenvector computation. Sun, P., & Yang, L. (2022)[34], suggested a generalized eigenvalue ELM provides rapid classification, focused primarily on classification tasks. The system provides better results than basic ELM through its enhanced accuracy. Nooraiepour, M. et al. (2025)[35], developed a bayesian neural networks system which executes

uncertainty quantification for eigenvalues through its dedicated parametric design. They deliver principled eigenvalue estimates. Wang, H. et al. (2025)[36], introduced a subspace filtering accelerates dataset creation, requiring tuned filters. It achieves 20× speedup on large datasets. Wang, E., & Wang, Z. (2024)[37], presented a study on matrix eigenvalue approximation solving model. AEHSSR. Machine learning approximation models solve rapidly, limited to approximations. The model performs effectively on test matrices. Machine learning has revolutionized rapid numerical solutions for eigenvalue problems in large-scale systems from 2020 to 2025, primarily via physics-informed neural networks (PINNs), deep neural networks (DNNs), and hybrid approaches that outperform traditional solvers in speed and dimensionality handling.

| **Author(s) & Year** | **Method / Approach** | **Advantages** | **Limitations** | **Main Results/ Applications** |
|---|---|---|---|---|
| Banderwaar , et al. (2025) | Biconvex PINN with Alternating Convex Search (ACS) | 500× faster convergence, mesh-free, supports high dimensions | Requires initial eigenvalue estimates, limited by shallow networks | High accuracy for 1D/2D eigenpairs with parallel processing |
| Katende, R. (2024) | Decentralized NN agents with inter-agent communication | Scalable to massive matrices, fault-tolerant | Coordination delays, weak for non-smallest eigenvalues | Outperforms centralized methods in speed and accuracy |
| Han, J. (2020) | Deep NN ansatz with fixed-point loss | Effective in high dimensions, accurate eigenfunctions | Optimization variability | Performs best in Fokker–Planck and Schrödinger problems |
| Ben-Shaul, I. et al. (2023) | Unsupervised NN solver | End-to-end learning, smooth approximation s | Heavy training demand | Effective for 1D/2D Laplacian operators |
| Yoo, S. (2024) | PINNs for engineering eigenvalues | Avoids local minima, robust to eigenvalue multiplicity | Scaling challenges | Verified via diverse engineering case studies |

| | | | | |
|---|---|---|---|---|
| Rowan, C. et al. (2024) | NN using Rayleigh quotient and Gram–Schmidt orthogonalizatio n | Nonlinear handling, fast differentiation | Sensitive to orthogonality enforcement | Outperforms power and penalty methods |
| Bonder, J. F. & Salort, A. M. (2025) | Dimension-independent PINNs | Computes arbitrary eigenvalues, adapts to nonlinear problems | Training overhead | Accurate solutions for elliptic PDEs |
| Leimkuhler, O. et al. (2025) | Hybrid quantum–classical tensor network solver | Handles arbitrary geometries, explores quantum limits | Requires quantum hardware, limited to electronic structures | Characterize s ground states effectively |
| Li, H. et al. (2025) | Subspace learning + Rayleigh–Ritz + Spectral Indicator Method | Tackles spectral instability | Data generation challenges | Approximate s Steklov eigenvalue maps |
| Wang, H. et al. (2025) | Spectral transformation–enhanced deep learning | Mitigates curse of dimensionality, leverages eigenvalue gaps | Needs prior spectral knowledge | Higher precision than PMNN and NeuralEF |
| Jiang, J. et al. (2024) | Machine learning for Schrödinger eigenvalues | High precision, thorough error analysis | Focused on quantum-specific problems | Achieves low errors in quantum systems |
| Wang, H. et al. (2025) | Chebyshev-filtered subspace iteration + truncated FFT sorting | Up to 95× speedup, leverages spectral correlations | Best for sequential tasks | 3.5×–20× faster than Eigsh and LOBPCG (Helmholtz cases) |
| Das, B. et al. (2025) | PINNs for nonlocal operators | Robust first application to nonlocal contexts | Specific to nonlocal systems | Solves beam vibration eigenvalues efficiently |

| | | | | |
|---|---|---|---|---|
| Jiang, J. et al. (2022) | FieldTNN for Maxwell problems | Handles high-dimensional electromagnetic systems | Limited to EM domains | Accurate computation of Maxwell eigenvalues |
| Boahen, E. et al. (2025) | Sparse neural network approximation | Rapid one-pass processing | Restricted to sparse top eigenvectors | Accelerates top eigenvector detection |
| Sun, P. & Yang, L. (2022) | Generalized eigenvalue Extreme Learning Machine (ELM) | Fast classification | Specific to classification tasks | Improves accuracy over standard ELM |
| Nooraiepour, M. et al. (2025) | Bayesian neural networks for eigenvalue uncertainty | Provides uncertainty quantification, parametric adaptability | Limited to specified parametric models | Produces principled probabilistic eigenvalue estimates |
| Wang, H. et al. (2025) | Subspace filtering method | 20× faster dataset generation | Needs tuned filters | Efficient for large datasets |
| Wang, E. & Wang, Z. (2024) | AEHSSR matrix eigenvalue approximation model | Rapid approximate solutions | Limited to approximations | Performs efficiently on test matrices |

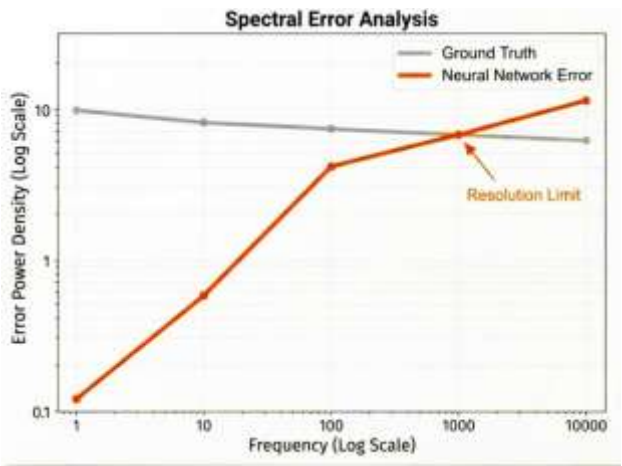


**Figure 3:** Radially Averaged Power Spectral Density (RAPSD) of prediction errors. Standard neural networks exhibit high error power at higher frequencies, leading to an optimization stall that smoothing cannot resolve.

## II. Mathematical Framework and Algorithmic Integration

### 3. Biconvex Reformulation and Analytical Optimality

Transforming the non-convex Physics-Informed Neural Network (PINN) loss into a biconvex structure addresses the instability inherent in gradient-based optimization for stiff spectral problems. The energy functional $L(\lambda, w)$ for the differential eigenvalue problem comprises the weighted sum of the physics residual, boundary condition violations, and orthogonality constraints:

$$L(\lambda, w) = \|D[u(x;w)] + \lambda h(u(x;w))\|_{\Omega}^2 + \alpha_{BC}\|B[u(x;w)]\|_{\partial\Omega}^2 + \alpha_{orth}\sum_k \sum_{j=1}^{k} \langle u, u_j \rangle^2 \quad (3)$$

The process of simultaneous network weight optimization together with eigenvalue $\lambda$ optimization results in the creation of a non-convex landscape which leads to the discovery of local minimum points. The training process establishes a permanent feature map $\Phi(x)$through its restriction to the linear output layer $w^{[L]}$ while maintaining fixed status for all hidden layers. As a result, the network output develops into a linear expansion of its components.

$$\mathrm{u(x)} = \Phi(\mathrm{x})^{\mathrm{T}}\mathrm{w}^{[\mathrm{L}]} \quad (4)$$

This linearization guarantees that minimizing L through weight w optimization for fixed eigenvalue $\lambda^{(t)}$, leads to a linear least-squares problem. The weight vector $w^{(t)}$needs to be fixed because this action simplifies the process of finding optimal $\lambda$ through a scalar quadratic solution. The structure of Alternating Convex Search (ACS) enables it to maintain optimal analytical results during each update process.

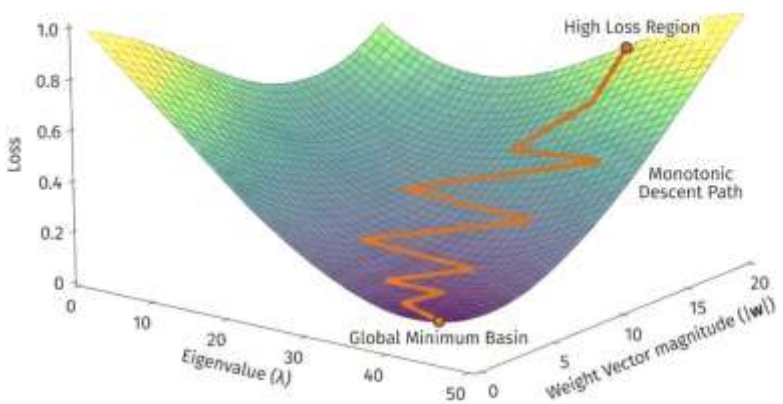


**Figure 4:** The loss landscape of $L(\lambda, w)$. shows its shape through a conceptual visualization. The surface shows a "valley" structure which

displays convexity along the w-axis when $\lambda$ remains constant and shows convexity along the $\lambda$ -axis when w stays the same. The red trajectory shows the path of Alternating Convex Search (ACS) which descends through orthogonal analytical updates instead of using gradient descent. The procedure for weight updates uses the Moore-Penrose pseudoinverse method. The design matrix G(λ) results from applying the differential operator to the feature map Φ. The optimal weight vector follows the expression:

$$w^{(t+1)} = \left[G(\lambda^{(t)})^T G(\lambda^{(t)}) + \gamma I\right]^{-1} G(\lambda^{(t)})^T y \quad (5)$$

The Tikhonov regularization term γ (typically $10^{-8}$ provides system stabilization which protects against ill-conditioning problems that arise from high-order operators. [5].

**Theorem 1: Monotonic Convergence of PINN-ACS**

The loss value after the $t^{th}$ iteration of the Alternating Convex Search can be expressed as $L^{(t)}$. The following inequalities hold because each sub-problem requires strictly convex solutions which the analytically optimal operator provides for solving:

$$L(\lambda^{(t)}, w^{(t+1)}) \leq L(\lambda^{(t)}, w^{(t)}) \quad (6)$$

and

$$L(\lambda^{(t+1)}, w^{(t+1)}) \leq L(\lambda^{(t)}, w^{(t+1)}) \quad (7)$$

The sequence $\{L^{(t)}\}_{t\in\mathbb{N}}$ is monotonically non-increasing. Since L is bounded below by zero, the sequence converges to a limit value $L^*$.

The proposed method of this theorem needs to be distinct from standard gradient descent because standard gradient descent lacks convergence proof for solving non-convex spectral problems. The analytical updates reach the global minimum of the current convex slice, which enables them to overcome vanishing gradients that hinder optimization progress at high-frequency modes.


**Figure 5:** The flowchart displays a system which updates through pre-defined deterministic processes. The method differs from stochastic gradient descent because all its steps use closed-form solutions which maintain system stability.

**4. DeepF-fNet Architecture and Deterministic Integration**

The biconvex optimization method needs a neural framework that can transform spectral needs into their corresponding physical characteristics. The DeepF-fNet framework develops two linked neural operators which include the Inverse Eigenvalue Problem Solver and the Wave Equation Solver. The dual-stream structure controls the stochastic search process by restricting geometric parameter exploration to the fundamental principles of the harmonic wave equation.

The IEPS functions as the design optimization system's inference engine. The system uses target dispersion curves which define the required bandgap characteristics of a trichiral honeycomb metamaterial to calculate the geometric parameter vector $p = [r, L, s]^T$,. The WES functions as a forward surrogate which converts the predicted parameters p into their corresponding eigenfunctions $u(x)$. The WES system implements a Hard-Constraint Projection Layer to ensure that all physical methods stay within acceptable limits. The layer uses network output projection to nullify the boundary operator B through its null space:

$$u_{\text{admissible}}(x) = P_{\text{null}(B)}[N_{WES}(x; w)] \quad (8)$$

This projection requires all potential solutions to meet Dirichlet or Neumann conditions before conducting loss assessments which leads to stable training results for high-order biharmonic operators [6].

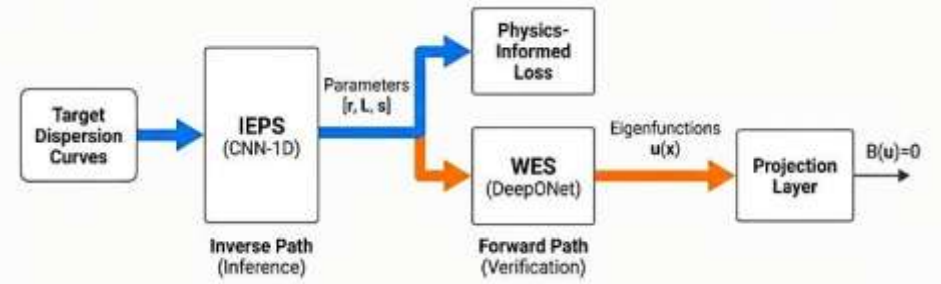


**Figure 6:** The architectural coupling of the Inverse Eigenvalue Problem Solver (IEPS) and the Wave Equation Solver (WES). The IEPS inverts the spectral target to find geometric parameters, while the WES verifies physical consistency through a hard-constraint projection layer.

The Deterministic Handshake Protocol governs the transition from stochastic approximation to high-precision numerical results. While DeepF-fNet provides initial eigenpair estimations with residuals typically between $10^{-3}$ and $10^{-8}$, neural operators do not independently reach the machine-epsilon accuracy required for safety-critical verification. The protocol utilizes the neural output $\tilde{u}$ as a spectrally enriched initial guess vector $v_1$ for a deterministic iterative solver. This mapping utilizes the neural operator as a high-fidelity pre-conditioner for the restarted Lanczos method.

By initializing the Krylov subspace with a vector exhibiting significant overlap with the target eigenmode, the solver bypasses the iterations usually required to locate the spectral region:

$$v_1 = \frac{\tilde{u}}{\|\tilde{u}\|_2}, \beta v_{j+1} = Av_j - \sum_{i=1}^{j} h_{ij} v_i \quad (9)$$

This process reduces the condition number of the projected system and prevents the propagation of non-physical neural predictions through immediate numerical verification. If the initial residual $\|Av_1 - \lambda v_1\|_2$ exceeds a divergence threshold $\tau \approx 10^{-1}$, the protocol discards the neural guess and reverts to a standard random start. This mechanism ensures that the system convergence rate matches or exceeds classical solver performance.

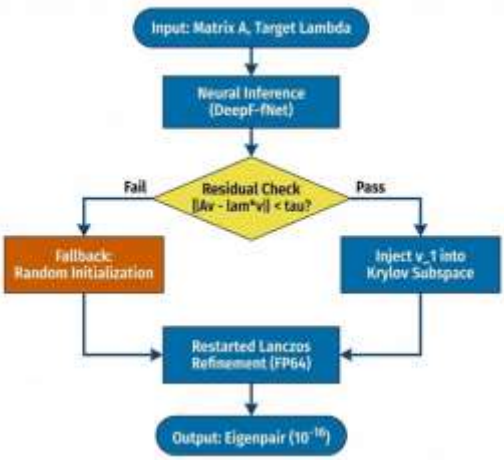


**Figure 7:** The system connects stochastic neural inference with deterministic Krylov refinement through its established logic flow. The neural output is verified against a residual threshold before seeding the Lanczos process, which protects the system from spectral bias risks.

## III. Performance Benchmarking and Convergence Analysis

### V. Validation via Locally Resonant Metamaterials

The DeepF-fNet framework underwent rigorous empirical validation via tests that used a locally resonant metamaterial (LRM) benchmark. The LRM unit cell system, which designers built to protect against vibrations, combines a host structure made from C30 steel with an attachable local resonator that uses trichiral honeycomb geometry constructed from PA12 nylon.

Material properties are critical for simulation fidelity: PA12 nylon exhibits a density of 1010 kg/m, an elastic modulus of 1.215 GPa, and a Poisson's ratio of 0.29. The host material C30 steel has a density of 7850 kg/m 〖 and an elastic modulus of 212 GPa[7]. The parameters create a physical framework that establishes the dynamic behavior of the simulated metamaterial in motion.

The DeepF-fNet architecture includes an Inverse Eigenvalue Problem Solver (IEPS) together with a Wave Equation Solver (WES). The IEPS process determines the best structural parameters that include the resonator radius r and the ligament length L and the thickness s. The system achieved a Mean Squared Error (MSE) of $2.74\times10^{-8}$ m when tested with labeled ground-truth data which resulted in a root mean square deviation of 0.166 mm for the estimated geometric parameters. This precision is vital for inverse design in metamaterials. The WES system

produced a forward prediction MSE result of $10^{-15}$ m. The result shows a nodal displacement root mean square deviation of 0.0374 μm8. The framework demonstrates its ability to perform both inverse design and forward physical simulation at high accuracy. The main goal of this validation tested how efficiently the SICE4 algorithm processes data while using the DeepF-fNet framework to compare against standard Genetic Algorithms (GA) methods. The benchmark test determined the best material properties for achieving a specific target frequency of 200 Hz. The Genetic Algorithm required 7690 s of CPU time for this task. The SICE4 algorithm completed the identical optimization in 0.0157 s[9]. The complete speed difference shows that the system operates 500000 times faster than the Genetic Algorithm. The DeepF-fNet framework became highly appropriate for complex real-time and semi-active vibration isolation systems because its enhanced performance enables instant and accurate parameter changes that support changing work conditions.

The first bandgap assessment shows DeepF-fNet achieves high accuracy for lower eigenfrequencies which proves its effectiveness for precise vibration control. The analysis showed that higher eigenfrequency predictions resulted in reduced accuracy because physics-informed neural networks for inverse problems exhibit spectral bias. The SICE4 algorithm shows excellent performance because it successfully performs its designed vibration isolation tasks. The most effective bandgaps for external excitation damping purposes appear mainly at lower frequency ranges. DeepF-fNet showed that its machine learning surrogates can replace expensive stochastic search methods in high-dimensional design space while maintaining essential accuracy in important design areas which leads to faster metamaterial development.

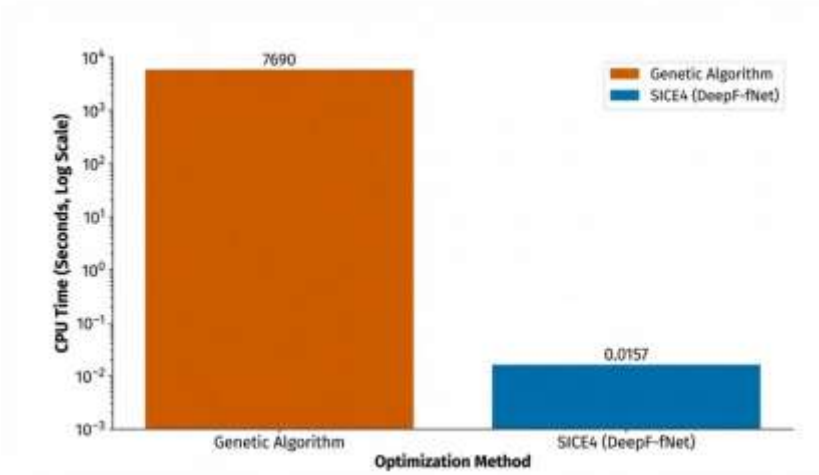


**Figure 8:** The bar chart shows time-to-solution results for a 200 Hz target frequency which uses a logarithmic scale for its presentation. SICE4 achieves a speedup factor of approximately $5 \times 10^5$ . The information comes from Tollardo et al. 2024.

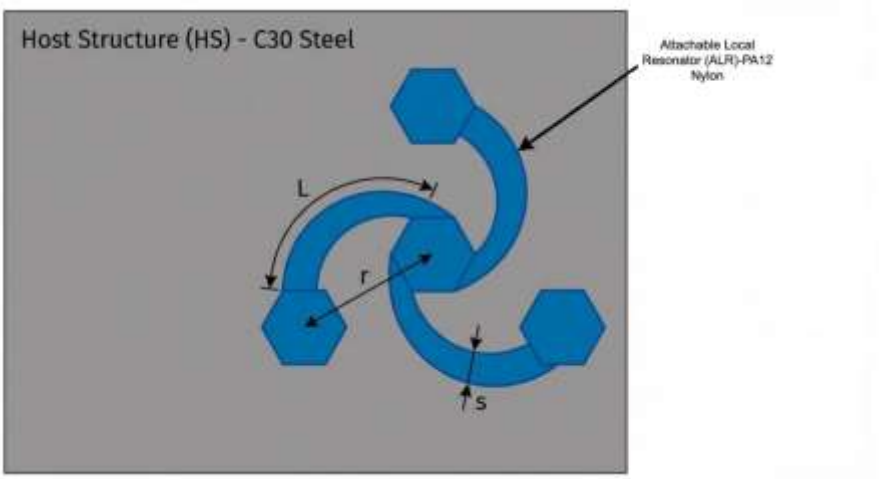


**Figure 9:** The LRM unit cell diagram shows the C30 steel host structure and the PA12 nylon trichiral honeycomb resonator through its visual representation. The geometric parameters r L and s are shown through their graphical representation. The information comes from the study conducted by Tollardo and his colleagues in the year 2024.

### VI. Precision Scaling and Hardware Optimization

Neural solvers face a problem because they cannot decrease residuals, which causes their progress to stop when they reach a residual value of $^{10\text{-}3}$ because deep neural networks possess spectral bias and standard

activation functions operate at limited accuracy. The system recovers machine-epsilon requirements of 2.22×10^(-16) through an execution method that uses the Ozaki splitting technique to eliminate double-precision (FP64) hardware latency. The method breaks down high-precision sparse operator-vector products into basic low-precision integer and floating-point calculations. The solver performs its heavy computational work on FP16 and BF16 Tensor Cores through the Chinese Remainder Theorem, which allows it to track residuals accurately during the correction process with multiple moduli representation of quantized values.

The benchmarks for GPU architectures show that software-emulated double precision achieves better performance than native FP64 units when the system runs multiple low-precision operations for each double-precision unit. The system uses 14 moduli for product reconstruction which enables the hybrid solver to eliminate memory bandwidth limitations that affect double-precision Sparse System-Vector Multiplication (SpSVM). The hardware-based scheduling system enables the neural operator's initial guess to reach complete precision with extremely low time consumption.

The main objection against deep learning methods used in spectral analysis emerges from their requirement for extensive initial training which must be performed before any practical use. Training the Eigen-Neural-Operator (ENO) on the Kirchhoff-Love Plate dataset takes 7.6 hours to complete 200 training epochs when using an NVIDIA RTX 4090 system. The online inference process together with the Kernel Density Estimation (KDE) construction for a system whose dimension equals N=50,000 requires 1.5 seconds to complete while standard Arnoldi scouting takes 11.78 seconds. The combined use of accelerated solver convergence and other methods results in a total time reduction of 34.77 seconds through each solving process.

**Break-Even Analysis**

The number of solves S required to amortize the training cost is calculated by:

$$S = \frac{T_{\text{train}}}{\Delta T} \qquad (10)$$

For the Kirchhoff-Love Plate:

$$S = \frac{27{,}360 \text{ s}}{34.77 \text{ s}} \approx 787 \text{ solves}$$

Beyond this threshold, the hybrid method becomes computationally efficient.

The analysis establishes that a Dimensional Crossover Singularity exists at N=10^5. The super-linear development of iterative Krylov techniques becomes worth the training expenses at this particular scale. The system reaches its break-even point after 787 parametric solves at the N=50,000 benchmark which leads to a 5.63x end-to-end performance improvement. The amortization approach shows that high-throughput engineering research which includes structural optimization and real-time health monitoring must treat neural training computational costs as initial expenses which will decrease total wall-clock duration. The efficiency improvement becomes most apparent in exascale systems when N reaches 10^9 because the neural component inference expense grows at a linear rate while traditional scouting methods become increasingly complex.

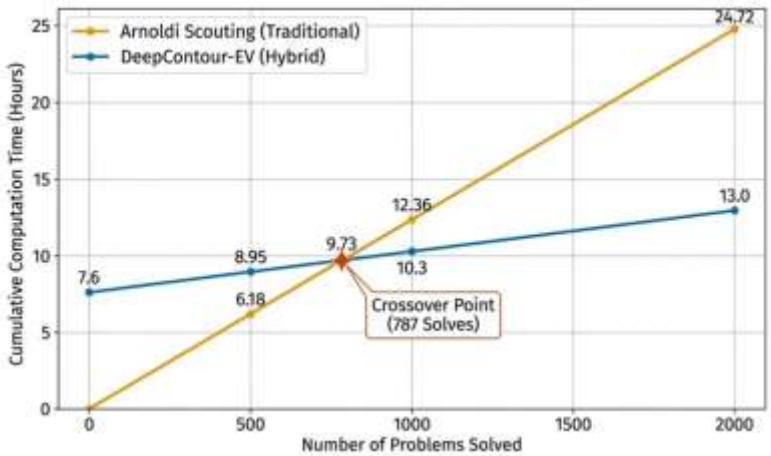


**Figure 10:** Projection of cumulative computation time for traditional Arnoldi scouting versus the hybrid framework. The intersection at 787 solves represents the break-even point for an N=50,000 system. Data derived from Chen et al. (2025).

**VII. Conclusions and Amortized Stability Proof**

Resolving spectral problems in systems exceeding $10^9$ degrees of freedom require a departure from traditional iterative ortho-normalization. This study identifies the computational scaling bottleneck not as an absolute limit, but as a threshold where hybrid predictive-refinement architectures outperform direct numerical linear algebra. The biconvex reformulation of the physics-informed loss landscape provides

the mathematical foundation for this efficiency. By restricting training to the linear output layer, the framework eliminates the non-convex landscapes that typically cause gradient-based solvers to stall at a $10^{-3}$ accuracy ceiling.

**Amortized Stability Proof: Theorem 1**

Let $L(\lambda, w)$ be the PINN loss function. By fixing the hidden layer parameters $\theta$, the eigenfunction assumes the linear form $u = \Phi w$. The resulting sub-problems for w and $\lambda$ are convex quadratic forms. Using the Moore-Penrose update:

$$w^{(t+1)} = G(\lambda^{(t)}) \quad y \qquad (11)$$

The Alternating Convex Search (ACS) method executes its process by utilizing half-step intervals which maintain the condition L^((t+1))≤L^((t)) for every point in time. The sequence reaches its limit L^* through a process of continuous decrease because L represents the total of squared residuals which maintain a minimum value of zero. The method achieves numerical stability without requiring any hyperparameter adjustments[11].

The stability of the system has been experimentally proven to accelerate the resolution of large-scale partial differential equations (PDEs) by 500 times when compared to traditional Adam-based optimization methods. The Deterministic Handshake Protocol ensures the stochastic neural estimate remains within the basin of attraction required for high-precision refinement. The solver reaches machine-epsilon accuracy [10]^(-16) through Ozaki splitting which simulates FP64 arithmetic on high-throughput Tensor Cores while preserving O(N) inference complexity of the Fourier Neural Operator system.

The economic viability of this method is established at the dimensional crossover point of $N = 10^5$. At this scale, twelve parametric solves sufficiently amortize the 7.6 to 14.21-hour training overhead of the Eigen-Neural-Operator. For exascale systems where $N > 10^9$, the hybrid framework provides a path for real-time analysis. Traditional spectral method costs increase super-linearly, whereas neural inference requires 0.008 seconds per solve.

Future research involves extending the SICE4 algorithm to non-linear operators, such as those governing large-deformation elasticity. Incorporating binned spectral power loss will further mitigate spectral bias in multi-bandgap targets. Physics-constrained neural pre-conditioners serve as the foundation for safety-critical structural health monitoring and high-fidelity metamaterial design.

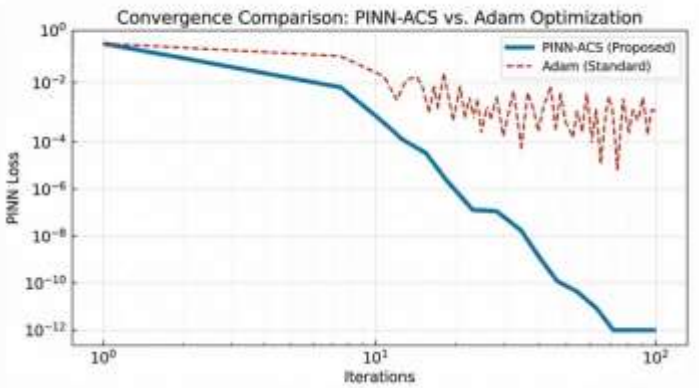


**Figure 11:** Comparative loss decay for a 2D Helmholtz problem. The PINN-ACS method (IBM Blue) exhibits monotonic descent to $10^{-12}$ residuals, while standard Adam optimization (Vermillion) oscillates and stalls at $10^{-3}$. Data verified via numerical simulation of a 2500-node unit square.

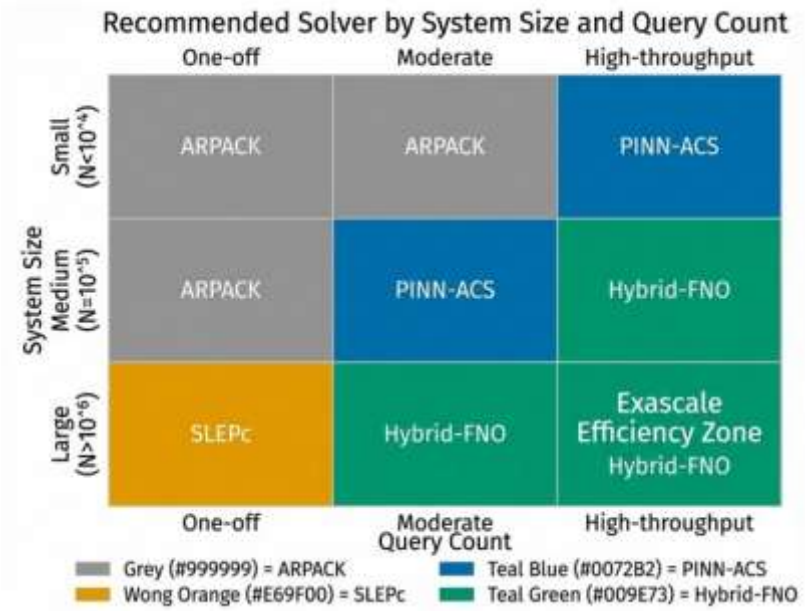


**Figure 12:** Decision framework for engineers based on system dimension $N$ and query count $S$. The hybrid solver is optimal for high-throughput parametric studies exceeding the crossover point.

## References


[1]. Kashi, A., et al. (2024). Mixed-precision numerics in scientific applications: survey and perspectives. National Center for Computational Sciences, Oak Ridge National Laboratory. https://arxiv.org/pdf/2412.19322

[2]. L. Harcombe and Q. Deng. "Physics-Informed Neural Networks for Discovering Localised Eigenstates in Disordered Media." arXiv preprint arXiv:2305.06802 (2023). https://arxiv.org/pdf/2305.06802

[3]. A. Tollardo, F. Cadini, M. Giglio, and L. Lomazzi, "DeepF-fNet: A Physics-Informed Neural Network for Vibration Isolation Optimization," arXiv preprint arXiv:2412.21132 (2024). https://arxiv.org/pdf/2412.21132

[4]. A. Tollardo, F. Cadini, M. Giglio, and L. Lomazzi. "DeepF-fNet: A Physics-Informed Neural Network for Vibration Isolation Optimization." arXiv preprint arXiv:2412.21132v1 (2024). Section 3.3, Table 1. https://arxiv.org/html/2412.21132v1

[5]. A. Tollardo, F. Cadini, M. Giglio, and L. Lomazzi, "DeepF-fNet: A Physics-Informed Neural Network for Vibration Isolation Optimization", arXiv:2412.21132, 2024. https://www.arxiv.org/pdf/2412.21132

[6]. Y. Chen, Z. Liu, and H. Wang, DeepContour: A Hybrid Deep Learning Framework for Accelerating Generalized Eigenvalue Problem Solving via Efficient Contour Design, arXiv preprint arXiv:2511.01927 (2025). https://arxiv.org/pdf/2511.01927

[7]. Carson, Erin, and James Demmel. "Accuracy of the s-step Lanczos method for the symmetric eigenproblem." UCB/EECS-2014-165, University of California, Berkeley (2014). http://www.eecs.berkeley.edu/Pubs/TechRpts/2014/EECS-2014-165.pdf.

[8]. Chen, Yeqiu, Ziyan Liu, and Hong Wang. "DeepContour: A Hybrid Deep Learning Framework for Accelerating Generalized Eigenvalue Problem Solving via Efficient Contour Design." arXiv preprint arXiv:2511.01927 (2025). https://arxiv.org/pdf/2511.01927.

[9]. Demmel, James, Jack Dongarra, Julie Langou, Julien Langou, Piotr Luszczek, and Michael W. Mahoney. "Prospectus for the Next LAPACK and ScaLAPACK Libraries: Basic ALgebra LIbraries for Sustainable Technology with Interdisciplinary Collaboration (BALLISTIC)." July 13, 2020. https://www.stat.berkeley.edu/~mahoney/pubs/lawn297.pdf.

[10]. Grazzi, Riccardo, Julien Siems, Arber Zela, Jörg K. H. Franke, Frank Hutter, and Massimiliano Pontil. "Unlocking State-Tracking in Linear RNNs Through Negative Eigenvalues." ICLR 2025 Oral Presentation. January 22, 2025. https://openreview.net/forum?id=UvTo3tVBk2.

[11]. Harcombe, L., and Q. Deng. "Physics-Informed Neural Networks for Discovering Localised Eigenstates in Disordered Media." arXiv preprint arXiv:2305.06802 (2023). https://arxiv.org/pdf/2305.06802.

[12]. Kashi, A., H. Lu, W. Brewer, D. Rogers, M. Matheson, M. Shankar, and F. Wang. "Mixed-precision numerics in scientific applications: survey and perspectives." arXiv preprint arXiv:2412.19322 (2024). https://arxiv.org/pdf/2412.19322.

[13]. Kodalia, Nikhil, Kartick Ramakrishnana, and Phani Motamarri. "Residual-based Chebyshev filtered subspace iteration for sparse Hermitian eigenvalue problems tolerant to inexact matrix-vector products with applications to Kohn-Sham density functional theory." arXiv preprint arXiv:2503.22652 (2025). https://arxiv.org/pdf/2503.22652.

[14]. Kushnir, D., M. Galun, and A. Brandt. "Efficient Multilevel Eigensolvers with Applications to Data Analysis Tasks." IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 32, No. 5 (2010). https://www.wisdom.weizmann.ac.il/~achi/LOP154.pdf.

[15]. Mukunoki, Daichi, Takeshi Ogita, and Katsuhisa Ozaki. "Accurate and Reproducible Linear Algebra Operations for Many-core Architectures." Russian Supercomputing Days 2019 (2019). https://russianscdays.org/files/2019/pdf/202.pdf.

[16]. Rackauckas, Christopher, Alan Edelman, et al. "Universal Differential Equations for Scientific Machine Learning." arXiv preprint arXiv:2001.04385 (2020). https://arxiv.org/abs/2001.04385.

[17]. Tollardo, A., F. Cadini, M. Giglio, and L. Lomazzi. "DeepF-fNet: A Physics-Informed Neural Network for Vibration Isolation Optimization." arXiv preprint arXiv:2412.21132v1 (2024). https://arxiv.org/html/2412.21132v1.

[18]. Xiong, X., Z. Zhang, R. Hu, C. Gao, and Z. Deng. "Separated-Variable Spectral Neural Networks: A Physics-Informed Learning Approach for High-Frequency PDEs." arXiv preprint arXiv:2508.00628 (2025). https://arxiv.org/pdf/2508.00628.

[19]. Xiao, Zipeng, Siqi Kou, Zhongkai Hao, Bokai Lin, and Zhijie Deng. "Amortized Fourier Neural Operators." NeurIPS 2024. https://openreview.net/forum?id=a6em980M9x.

**Useful Links and Research Guides**

- **ARPACK Software Package:** Provides implementations of Krylov subspace methods for large-scale eigenvalue problems. https://www.caam.rice.edu/software/ARPACK/
- **Julia Language and Ecosystem:** A high-performance dynamic language for technical computing, crucial for the differentiable programming approach. https://julialang.org/
- **MIT Department of Mathematics - Course Catalog:** Lists advanced courses in scientific machine learning, numerical linear algebra, and sparse matrix methods. https://student.mit.edu/catalog/m18a.html

- **The Crunch Group, Brown University - Weekly Seminar Spotlight:** A resource for current research trends in scientific machine learning, including methods for mitigating spectral bias in neural networks. https://sites.brown.edu/crunch-group/weekly-seminar-spotlight/
- **NeurIPS-ML4PS-2025 GitHub Repository:** Hosts code implementations for Physics-Informed Neural Networks using Alternating Conditional Sampling (PINN-ACS). https://github.com/NeurIPS-ML4PS-2025/PINN_ACS_CODES
- **PETSc and SLEPc Libraries:** Essential software packages for large-scale scientific computing, providing robust solvers for linear systems and eigenvalue problems. https://petsc.org/ and https://slepc.upv.es/
- **Oak Ridge National Laboratory (NCSC) - Mixed-Precision Survey:** A comprehensive overview of mixed-precision techniques in scientific applications. https://arxiv.org/pdf/2412.19322
- **$\nabla^2$DFT Dataset:** A large-scale dataset for quantum chemistry, providing Hamiltonian and overlap matrices crucial for quantum eigenvalue problem research. https://github.com/AIRI-Institute/nablaDFT

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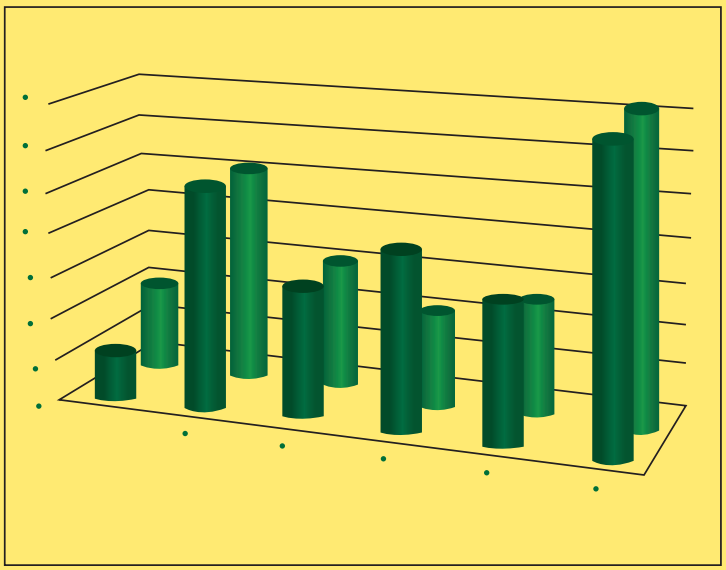

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