ENDPOINT-INFLATED POISSON DISTRIBUTION

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Abstract

This article is concerned with Endpoint-Inflated Poisson distribution for modeling count data with large frequencies of zeros and right-endpoint m. The proposed model is said to be inflated since it allows for positive probability mass at some points (zero and m). The model is a mixture of three distributions, a degenerate distribution with a unit point mass at zero, a degenerate distribution with a unit point mass at m which assign higher probabilities to 0 and m and the other counts are defined by the Poisson distribution. Some of its properties is discussed. The maximum likelihood and moment approaches are utilized to derive point estimates and confidence intervals for the distributional parameters. The elements of the Hessian matrix, the Fisher information matrix and the variance-covariance matrix of the maximum likelihood estimators are derived. An application using two real data sets is presented. The performance of the model and its sub-models is assessed using goodness of fit test and different information criteria.

Keywords: Count data; Poisson distribution; Zero-Inflated Poisson distribution; Endpoint-inflated Poisson distribution; Zero- one Inflated Poisson distribution; Maximum likelihood estimators; Moment estimators.

1. Introduction

Many studies in different areas involve nonnegative integer values. The Poisson models are the most used tools for modeling count data. In practice, however, count data are often over dispersed, the variance can be greater than the average value. One frequent manifestation of over dispersion is that the incidence of zero counts is greater than expected for the Poisson distribution. Motivated by this fact, some studies have focused on inflated distributions for modeling count data with large frequencies of zeros that cannot be explained by models based on standard distributional assumptions. Inflated distributions can be thought as finite mixture distributions which involve a finite number of components to deal with the nature of the data. Mixture distributions arise when each distribution separately cannot describe the data. Some of these studies are interested in zero-inflated and others are interested in zero-and-one inflated families of models.

Zero-inflated distributions have been developed to fit count data with large frequencies of zeros that cannot be explained by models based on standard distributional assumptions. Such data are common in many fields including medicine, public health studies, epidemiology, ecology, sociology, psychology, econometrics, agriculture, engineering, manufacturing, and road safety. This method is based on an assumption that the random variable is generated by a mixture of two distributions, one is the discrete distribution and a degenerate distribution with a unit point mass at zero [see Mullahy (1986) developed zero inflated family of models, Lampert (1992) extended

zero-inflated Poisson (ZIP) distribution]. Many studies build regression models based on Zero-inflated distributions to clarify the relation between the covariates and the response variable. [see Lampert (1992) used a parametric ZIP regression model to study the effects of covariates with parameters of interest via appropriate link functions, Ridout et al. (2001) derived a score test for testing a ZIP regression model against zero-inflated negative binomial (ZINB) alternatives which the non-zero part of the count data is over dispersed and another distribution such as ZINB may be more appropriate than ZIP, Diop and Dupuy (2014) developed zero inflated Bernoulli (ZIBER) regression model to fit binary data that contain too many zeros. Fitriani et al. (2019) presented Simulation on the ZINB to model over dispersed Poisson distributed data, Diallo et al. (2019) presented estimation in zero-inflated binomial (ZIB) regression with missing covariates].

Zero-and-one inflated distributions have been developed to fit count data with excess zeros and ones simultaneously. For example, most car owners will claim zero or one time in the insurance period; one may be infected by some virus for at most one-time due to the generation of corresponding antibodies once after the infection, so the count of one can be underestimated by zero-inflated models. There are many ways to build zero-and-one inflated models and one of these methods is based on an assumption that the random variable is generated by a mixture of three distributions, one is the discrete distribution, a degenerate distribution with a unit point mass at zero, and a degenerate distribution with a unit point mass at one [see Edwin (2014) considered zero-one inflated geometric (ZOIG) distribution in analysis of a real life. Alshkaki (2016) introduced zero-and-one inflated power series distributions, Poisson, binomial, negative binomial, geometric and logarithmic series distributions. Alshkaki (2016) discussed properties and parameters estimators of zero-and-one inflated Poisson (ZOIP) distribution. Alshkaki (2016) provided mathematical properties of zero-one inflated logarithmic series (ZOILS) distribution. Alshkaki (2016) provided mathematical properties of zero-one inflated negative binomial (ZOINB) and zero-one inflated binomial (ZOIB) distributions. Zhang et al. (2016) studied the likelihood based ZOIP model without covariates. Tang et al. (2017) studied the statistical inference for (ZOIP) distribution. Liu et al. (2018) derived the objective Bayesian estimation of ZOIP model, Alshkaki (2019) drive a combined estimation method to estimate the parameters of the (ZOINB) distribution, Tlhaloganyang et al. (2019) derived Structural properties of zero-one-inflated negative-binomial crack (ZOINBCR) distribution].To investigate the relation between the covariates and the response variable, many studies build regression models based on zero-and-one inflated distributions [see Deng et al. (2015) introduced generalized endpoint-inflated binomial model, Liu et al. (2018) introduced zero-and-one inflated Poisson regression model].

In this article, endpoint-inflated model was developed to fit count data to handle variability from both excessive zeros and excessive right-endpoint m compared with other observations in the data .Such data are common in many fields including psychological, social, and public health related research. For example, many patients go to the cosmetology many times when others never visit; the number of days that individuals employ may be zero due to unemployment and may be many days; patients may be infected by the virus and have not received any doses of prescription

medication for lake of detection while others have received multiple doses for early detection; the number of days people with psychiatric problems spent in hospitals exceeds months while others are not fully hospitalized. The proposed model is suggested based on the strategy of inclusion an extra parameter, which is an extension of zero- inflated models through addition of the right-endpoint inflation parameter to provide alternative distributions for modeling count data that is found to be characterized by excessive zero and excessive right-endpoint counts. The model is based on an assumption that the random variable is generated by a mixture of three distributions, one is the discrete distribution, a degenerate distribution with a unit point mass at zero, and a degenerate distribution with a unit point mass at m, so the model is said to be inflated since it allows for positive probability mass at some points [zero and m].

The endpoint-inflated Poisson distribution is a mixture of three distributions, a degenerate distribution with a unit point mass at zero, a degenerate distribution with a unit point mass at m which assign higher probabilities to zero and m and the other counts are defined by the Poisson distribution.This article unfolds as follows; Section 2 presents the endpoint-inflated Poisson distribution. The main properties of the distribution such as the mean, variance, moment generating function, and the probability generating function are obtained in Section 3. Section 4 discusses the maximum likelihood estimators. The elements of the Hessian matrix, the Fisher information matrix and the variance-covariance matrix of the maximum likelihood estimators are derived in Section 5. In Section 6 the moment estimators are presented. Estimation of the parameters of the endpoint-inflated Poisson distribution and its sub-models as well as fitting their frequencies were presented in Section 7 using two different sets of data representing; the number of infected cases with Covid 19 in Africa data, the number of weekly worked days in Egypt data. Finally, some concluding remarks were given in section 8.

2. Endpoint-Inflated Poisson Distribution

Let Y be a discrete random variable has an endpoint-inflated Poisson distribution, denoted by $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$, the model can be expressed as a mixture of three probability functions as follows:

$$f(y; \varphi_0, \varphi_1, \lambda) = \varphi_o f_1(y) + \varphi_1 f_2(y) + \varphi_2 f_3(y), \quad (1)$$

where

$$f_1(y) = p\,(y = 0) = 1, \quad for \quad y = 0. \quad (2)$$

is a degenerate probability function with a unit point mass at zero,

$$f_2(y) = \mathrm{p}\,(\mathrm{y} = m) = 1, \quad \text{for } \mathrm{y} = m. \quad (3)$$

is a degenerate probability function with a unit point mass at m,

$$f_3(y) = \frac{e^{-\lambda}\lambda^y}{y!}, \text{ for } \quad y = 0,1,2, \ldots \quad , \quad \lambda > 0 \quad (4)$$

is the probability mass function (pmf) of the Poisson distribution, denoted by $\mathrm{PD}(\lambda)$, $\varphi_o \in [0,1]$, $\varphi_1 \in [0,1]$ denote the unknown proportion for incorporating extra zeros and extra right-endpoint m, respectively and $\varphi_2 = 1 - \varphi_0 - \varphi_1 \in [0,1]$.

then

$$f(y;\varphi_0,\varphi_1,\lambda) = \begin{cases} \varphi_o + \varphi_2 e^{-\lambda} & \text{if } y = 0, \\ \varphi_2 \dfrac{e^{-\lambda}\lambda^y}{y!}, & \text{if } y = 1,2,\dots,m-1,m+1, \\ \varphi_1 + \varphi_2 \dfrac{e^{-\lambda}\lambda^m}{m!} & \text{if } y = m, \end{cases} \qquad (5)$$

The model (5) is a three-component mixture model with two degenerate distributions at $y = 0$ and $y = m$, and the $\mathrm{PD}(\lambda)$. Figure 1 shows some different $\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$ probability mass functions along with the corresponding values of $(\varphi_0,\varphi_1,\lambda)$. It is noteworthy that the probability functions can display different shapes depending on the values of the three parameters. In particular, when $\varphi_0 = 0$, the $\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$ in (1) is reduced to *right-endpoint inflated Poisson* distribution, denoted by $\mathrm{REIP}\ (\varphi_1,\lambda)$ when $\varphi_1 = 0$, the $\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$ in (1) is reduced to *zero inflated Poisson* distribution, denoted by $\mathrm{ZIP}\ (\varphi_0,\lambda)$ which is proposed by Lampert (1992), when $\varphi_0 = \varphi_1 = 0$, the $\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$ in (1) becomes the $\mathrm{PD}(\lambda)$ in (4). Also, the subset models can accommodate, inflation-inflation $(\varphi_0 > 0, \varphi_1 > 0)$, inflation- deflation $(\varphi_0 > 0, \varphi_1 = 0)$, deflation- inflation $(\varphi_0 = 0, \varphi_1 > 0)$. However, the model can also be used to fit the data with zero deflation $(\varphi_0 = 0)$ and m deflation $(\varphi_1 = 0)$.

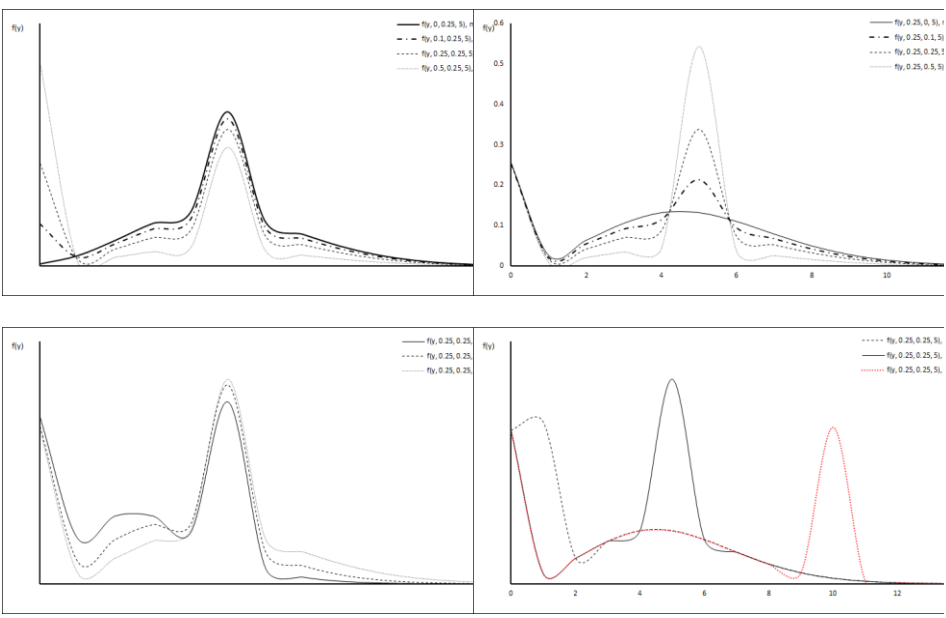

Figure 1 Endpoint-inflated Poisson Probability Mass Functions for different combination of $(\varphi_0, \varphi_1, \lambda)$ and m

The *cumulative distribution function* (cdf) of the $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$ is given as:
$F(y; \varphi_0, \varphi_1, \lambda) = P(Y \leq y)$

$$= \sum_{Y \leq y} f(y; \varphi_0, \varphi_1, \lambda), \qquad y = 0, 1, 2, \ldots. \tag{6}$$

3. Some Properties of the Endpoint-Inflated Poisson Distribution

The r[th] moment about the origin of the random variable Y can be obtained as follows:

$$E(Y^r) = \varphi_1 m^r + \varphi_2 \sum_{y=0}^{\infty} y^r \frac{e^{-\lambda}\lambda^y}{y!}, \quad r = 1, 2, \ldots \tag{7}$$

The *mean* and *variance* respectively, are given by:

$$E(Y) = \varphi_1 m + \varphi_2\, \lambda, \tag{8}$$

$$V(Y) = \varphi_1 m^2 + \varphi_2(\lambda^2 + \lambda) - (\varphi_1 m + \varphi_2\, \lambda),^2 \tag{9}$$

The *moment generating function* and the *probability generating function*, respectively, are given by

$$M_y(t) = \varphi_o + \varphi_1 e^{mt} + \varphi_2\, e^{\lambda(e^t - 1)}. \tag{10}$$

$$G_y(t) = \varphi_o + \varphi_1 t^m + \varphi_2\, e^{\lambda(t-1)} \tag{11}$$

By substituting $\varphi_o = 0$, $\varphi_1 = 0$ and $\varphi_0 = \varphi_1 = 0$, respectively, the main properties of the sub-models, $\mathrm{REIP}\ (\varphi_1, \lambda)$, $\mathrm{ZIP}(\varphi_0, \lambda)$ and $\mathrm{PD}(\lambda)$,respectively, can be obtained as special cases of the main properties of the $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$.

4. The Maximum Likelihood Estimators of the parameters

The *maximum likelihood estimation* (MLE) method is used to estimate the parameters of the $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$.
Let $y_1, \ldots, y_n$ be a random sample of size n drown from $f(y; \varphi_0, \varphi_1, \lambda)$ in (5).
The likelihood function of the observed sample is given by:

$$L\left(\underline{\theta}; \underline{y}\right) = \prod_{i=1}^{n} f(y_i; \varphi_0, \varphi_1, \lambda), \tag{12}$$

where $\underline{\theta} = (\varphi_0, \varphi_1, \lambda)$.
The likelihood function of $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$ is derived by substituting (5), in (12).

$$L\left(\underline{\theta}; \underline{y}\right) = \left[\varphi_o + \varphi_2 e^{-\lambda}\right]^{I_o} \left[\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}\right]^{I_1} \prod_{i=1}^{I_2} \left(\varphi_2 \frac{e^{-\lambda}\lambda^y}{y_i!}\right). \tag{13}$$

where

$I_o = I_o(y) = \#\{i: y_i = 0\}$,
$I_1 = I_1(y) = \#\{i: y_i = m\}_1$,

and

$I_2 = n - I_o - I_1.$

Here $\sharp\, \mathbb{x}$ is used to denote the number of elements of the set $\mathbb{x}$.

The natural logarithm of (13) can be obtained as follows:

$$\ell(\underline{\theta}) = ln\, L\left(\underline{\theta}; \underline{y}\right)$$
$$= I_o\, ln\left(\varphi_o + \varphi_2 e^{-\lambda}\right) + I_1 ln\left(\varphi_1 + \varphi_2 e^{-\lambda}\frac{\lambda^m}{m!}\right) + I_2 ln(\varphi_2) - I_2\lambda$$
$$+ \sum_{i=1}^{I_2} y_i\ ln(\lambda)$$
$$- \sum_{i=1}^{I_2} ln\, y_i!. \qquad (14)$$

The elements of the score vector for φ_0, φ_1 and λ can be obtained by taking the first partial derivatives of the log likelihood function (14) with respect to the unknown parameters $\underline{\theta} = (\varphi_0, \varphi_1, \lambda)$, as follows:

$$\frac{\partial \ell(\underline{\theta})}{\partial \varphi_0} = \frac{\left(1 - e^{-\lambda}\right)}{\varphi_o + \varphi_2 e^{-\lambda}} I_o - \frac{\left(e^{-\lambda}\frac{\lambda^m}{m!}\right)}{\varphi_1 + \varphi_2 e^{-\lambda}\frac{\lambda^m}{m!}} I_1 - \frac{1}{\varphi_2} I_2, \qquad (15)$$

$$\frac{\partial \ell(\underline{\theta})}{\partial \varphi_1} = -\frac{e^{-\lambda}}{\varphi_o + \varphi_2 e^{-\lambda}} I_o + \frac{\left(1 - e^{-\lambda}\frac{\lambda^m}{m!}\right)}{\varphi_1 + \varphi_2 e^{-\lambda}\frac{\lambda^m}{m!}} I_1 - \frac{1}{\varphi_2} I_2, \qquad (16)$$

$$\frac{\partial \ell(\underline{\theta})}{\partial \lambda} = -\frac{\varphi_2 e^{-\lambda}}{\varphi_o + \varphi_2 e^{-\lambda}} I_o + \frac{\varphi_2 e^{-\lambda}\left(\frac{m\lambda^{m-1}}{m!} - \frac{\lambda^m}{m!}\right)}{\varphi_1 + \varphi_2 e^{-\lambda}\frac{\lambda^m}{m!}} I_1 - I_2 + \frac{\sum_{i=1}^{I_2} y_i}{\lambda} \qquad (17)$$

From (15), (16) and (17), the score vector for φ_0, φ_1 and λ can be written as follows:

$$U(\underline{\theta}) = \left(\frac{\partial \ell(\underline{\theta})}{\partial \varphi_0}, \frac{\partial \ell(\underline{\theta})}{\partial \varphi_1}, \frac{\partial \ell(\underline{\theta})}{\partial \lambda}\right)^T.$$

The ML estimators of $\varphi_0, \varphi_1\ and\ \lambda$ can be obtained as the solution of the nonlinear system [see Alshkaki (2016)].

$$\left(\frac{\partial \ell(\underline{\theta})}{\partial \varphi_0}, \frac{\partial \ell(\underline{\theta})}{\partial \varphi_1}, \frac{\partial \ell(\underline{\theta})}{\partial \lambda}\right)^T = \underline{0}.$$

let

$$p_0 = \varphi_o + \varphi_2 e^{-\lambda}, \qquad (18)$$

$$p_1 = \varphi_1 + \varphi_2 e^{-\lambda}\frac{\lambda^m}{m!}. \qquad (19)$$

and they can be replaced by their sample relative frequencies, i.e. their sample estimates which are the proportion of zeros and the proportion of ms in the sample as follows:

$$\widehat{p_0} = \frac{I_o}{n}, \tag{20}$$

$$\widehat{p_1} = \frac{I_1}{n}. \tag{21}$$

By substituting (18) to (21) in (15) and setting to zero, φ_2 can be estimated as follows:

$$\hat{\varphi}_2 = \frac{I_2}{n\left(1 - e^{-\lambda} - e^{-\lambda}\frac{\lambda^m}{m!}\right)}. \tag{22}$$

By substituting (18) to (21) in (17), λ can be estimated using:

$$\mathcal{A}(\hat{\lambda}) = 0, \tag{23}$$

$$\mathcal{A}(\hat{\lambda}) = \left(\sum_{i=1}^{I_2} y_i - I_2\hat{\lambda}\right)\left(e^{\hat{\lambda}} - 1 - \frac{\hat{\lambda}^m}{m!}\right) + \hat{\lambda}\left(-1 + \frac{\hat{\lambda}^{m-1}}{(m-1)!} - \frac{\hat{\lambda}^m}{m!}\right). \tag{24}$$

Hence, (24) can be solved by any numerical procedure, to obtain $\widehat{\lambda}$ numerically, Similarly, using (18) to (24), φ_o and φ_1 can be estimated respectively as follows:

$$\hat{\varphi}_o = \frac{1}{n}\left[I_o - \frac{I_2}{e^{\widehat{\lambda}} - 1 - \frac{\widehat{\lambda}^m}{m!}}\right], \tag{25}$$

and

$$\hat{\varphi}_1 = \frac{1}{n}\left[I_1 - \frac{\frac{\widehat{\lambda}^m}{m!}I_2}{e^{\widehat{\lambda}} - 1 - \frac{\widehat{\lambda}^m}{m!}}\right]. \tag{26}$$

The maximum likelihood (ML) estimators for φ_1, λ of $\mathrm{REIP}(\varphi_1, \lambda)$, φ_0, λ of $\mathrm{ZIP}\ (\varphi_0, \lambda)$, and λ of $\mathrm{PD}(\lambda)$, can be obtained by substituting $I_0 = \varphi_0 = 0$, $I_1 = \varphi_1 = 0$ and $I_0 = \varphi_0 = I_1 = \varphi_1 = 0$, respectively in (14) and taking the partial derivatives with respect to the unknown parameters $\underline{\theta} = (\varphi_1, \lambda)$, $\underline{\theta} = (\varphi_0, \lambda)$ and λ, respectively, and following the same steps as before.

5. The variance-covariance matrix

The variance-covariance matrix of the ML estimators of the $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$, is the inverse of Fisher information matrix, the elements of the Fisher information matrix can be obtained by taking the negative expectation of the Hessian matrix.

The elements of the Hessian matrix of the ML estimators of the $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$, are obtained by taking the second derivatives of the natural logarithm of the likelihood function, $\ell(\underline{\vartheta})$ in (14) with respect to the unknown parameters, $\underline{\theta} = (\varphi_0, \varphi_1, \lambda)$, as follows [see Appendix A]:

$$\left[J(\underline{\theta})\right]_{i,j} = \left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ell(\underline{\theta})\right]\Bigg|_{\widehat{\theta}_i,\widehat{\theta}_j,}, \qquad i,j = 1,2,3.$$

the Hessian matrix can be written as follows:

$$J(\underline{\theta}) = \begin{pmatrix} J_{\varphi_o\varphi_o} & J_{\varphi_o\varphi_1} & J_{\varphi_o\lambda} \\ J_{\varphi_1\varphi_0} & J_{\varphi_1\varphi_1} & J_{\varphi_1\lambda} \\ J_{\lambda\varphi_0} & J_{\lambda\varphi_1} & J_{\lambda\lambda} \end{pmatrix}. \qquad (27)$$

The elements of the two 2x2 Hessian matrices of the ML estimators for $\mathrm{REIP}(\varphi_1,\lambda)$ and $\mathrm{ZIP}(\varphi_0,\lambda)$can be obtained as special cases of (27) when $I_0 = \varphi_0 = 0$ and $I_1 = \varphi_1 = 0$,respectively.

The elements of the Fisher information matrix of the ML estimators of the $\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$,are obtained by taking the negative expectation of the Hessian matrix (27) as follows [see Appendix B]:

$$\left[\mathrm{K}(\underline{\theta})\right]_{i,j} = -E\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ell(\underline{\theta})\right]\Bigg|_{\widehat{\vartheta}_i,\widehat{\vartheta}_j,},$$

Where: $i,j = 1,2,3.$ and $\underline{\theta} = (\varphi_0,\varphi_1,\lambda)$ the Fisher information matrix can be written as follows:

$$\mathrm{K}(\underline{\theta}) = \begin{pmatrix} \mathrm{K}_{\varphi_o\varphi_o} & \mathrm{K}_{\varphi_o\varphi_1} & \mathrm{K}_{\varphi_o\lambda} \\ \mathrm{K}_{\varphi_1\varphi_0} & \mathrm{K}_{\varphi_1\varphi_1} & \mathrm{K}_{\varphi_1\lambda} \\ \mathrm{K}_{\lambda\varphi_0} & \mathrm{K}_{\lambda\varphi_1} & \mathrm{K}_{\lambda\lambda} \end{pmatrix}. \qquad (28)$$

The elements of the two 2x2 Fisher information matrices of the ML estimators for $\mathrm{REIP}(\varphi_1,\lambda)$ and $\mathrm{ZIP}(\varphi_0,\lambda)$can be obtained as special cases of (28) when $I_0 = \varphi_0 = 0$ and $I_1 = \varphi_1 = 0$,respectively.

The variance-covariance matrix of the ML estimators of the $\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$, is the inverse of Fisher information matrix (28), can be obtained as follows [see Appendix c]:

$$\mathrm{K}(\underline{\theta})^{-1} = \frac{1}{\left|\mathrm{K}(\underline{\theta})\right|} adj\ \mathrm{K}(\underline{\theta}), \qquad (29)$$

where

$\left|\mathrm{K}(\underline{\theta})\right|$ is the determinant of $\mathrm{K}(\underline{\theta})$, (30)

and

$adj\ \mathrm{K}(\underline{\theta})$ is the adjoint of $\mathrm{K}(\underline{\theta})$. (31)

the variance-covariance can be written as follows:

$$k(\theta)^{-1} = \begin{pmatrix} K^{\varphi_o\varphi_o} & K^{\varphi_o\varphi_1} & K^{\varphi_o\lambda} \\ K^{\varphi_1\varphi_0} & K^{\varphi_1\varphi_1} & K^{\varphi_1\lambda} \\ K^{\lambda\varphi_o} & K^{\lambda\varphi_1} & K^{\lambda\lambda} \end{pmatrix}, \qquad (32)$$

The diagonal elements, k^{ll} of the variance-covariance matrix, $k(\theta)^{-1}$ in (32) are the variance of the ML estimators, $(\hat{\varphi}_0,\hat{\varphi}_1,\hat{\lambda})$ and the square roots of the diagonal elements of the variance-covariance matrix, are the standard errors of the ML estimators, $(\hat{\varphi}_0,\hat{\varphi}_1,\hat{\lambda})$.

It can be noticed that, the elements of the two, 2×2 variance-covariance matrices for $\mathrm{REIP}(\varphi_1, \lambda)$ and $\mathrm{ZIP}(\varphi_0, \lambda)$, can be obtained as special cases of (32) when $I_0 = \varphi_0 = 0$ and $I_1 = \varphi_1 = 0$, respectively, with diagonal elements of each are the variance of the ML estimators, $(\hat{\varphi}_1, \hat{\lambda})$ and $(\hat{\varphi}_0, \hat{\lambda})$ and the square roots of the diagonal elements are the standard errors of the ML estimators, $(\hat{\varphi}_1, \hat{\lambda})$ and $(\hat{\varphi}_0, \hat{\lambda})$ of the $\mathrm{ZIP}(\varphi_0, \lambda)$ and $\mathrm{REIP}(\varphi_1, \lambda)$, respectively.

The variance of the ML estimator, $(\hat{\lambda})$ of the $\mathrm{PD}(\lambda)$ can be obtained as follows:

$$\mathrm{V}(\hat{\lambda}) = \frac{1}{-\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial \lambda^2}\right)} = \frac{\hat{\lambda}}{n}, \qquad (33)$$

then, the standard error of the ML estimator, $\hat{\lambda}$ is the square root of this variance.

Thus, $(1-\alpha)100\%$ asymptotic confidence intervals (CIs) of $\hat{\varphi}_0, \hat{\varphi}_1$ and $\hat{\lambda}$ can be obtained as follows:

$$\hat{\theta} \pm z_{\frac{\alpha}{2}} (k^{ll})^{\frac{1}{2}}, \qquad (34)$$

$\hat{\theta} = (\hat{\varphi}_0, \hat{\varphi}_1, \hat{\lambda})$, and z_α represent the α^{th} quantile of the $N(0,1)$ distribution.

6. The Moment Estimators of the parameters

The moment estimation (ME) method is used to estimate the parameters of the $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$. The first three distribution moments about the origin for the $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$ can be found to be,

$\mu_1' = \varphi_1 m + \varphi_2 \lambda$,

$\mu_2' = \varphi_1 m^2 + \varphi_2 \lambda(1+\lambda)$,

and

$\mu_3' = \varphi_1 m^3 + \varphi_2 \lambda(1 + 3\lambda + \lambda^2)$.

Let $y_1, y_2, ,,,, y_n$ be a random sample from $f(y; \varphi_0, \varphi_1, \lambda)$ in (5), and let,

$$M_{\mathrm{r}}' = \frac{\sum_{i=1}^{n} y^r k_i}{\sum_{i=1}^{n} k_i}, \qquad \mathrm{r} = 1,2,3 \qquad (35)$$

be their sample moments about the origin, then solving the following simultaneous equations:

$$M_1' = \varphi_1 m + \varphi_2 \lambda, \qquad (36)$$

$$M_2' = \varphi_1 m^2 + \varphi_2 \lambda(1+\lambda), \qquad (37)$$

and

$$M_3' = \varphi_1 m^3 + \varphi_2 \lambda(1 + 3\lambda + \lambda^2). \qquad (38)$$

for φ_o , φ_1 and λ give the following ME for these parameters:

$$\hat{\varphi}_o = \frac{1}{2(-1+m)^2 m(M_2' - M_3' - M_1'm + M_2'm)}\Big(-2M_1'M_2' + {M_2'}^2 + 2M_1'M_3' - 2M_2'M_3' + {M_3'}^2 + m\big(2{M_1'}^2 + 2M_2' + 6{M_2'}^2 - 2M_3' - 2M_1'M_3' - 4M_2'M_3'\big) - m^2\big(2M_1' + 3{M_1'}^2 + 2M_2' + 8M_1'M_2' - 2{M_2'}^2 - 4M_3' - 4M_1'M_3'\big) + m^3\big(4M_1' + 6{M_1'}^2 - 2M_2' - 2M_1'M_2' - 2M_3'\big) - m^4\big(2M_1' + {M_1'}^2 - 2M_2'\big) + \mathbb{Q}(-M_2' + M_3' + M_1'm - 2M_2'm + M_1'm^2)\Big), \quad (39)$$

$$\hat{\varphi}_1 = \frac{2M_1' - M_2' + M_3' - m(M_1' - 2M_2') + M_1'm^2 + \mathbb{Q}}{2(1-m)m}, \quad (40)$$

and

$$\hat{\lambda} = \frac{3M_2' - M_3' - 3M_1'm + M_1'm^2 + \mathbb{Q}}{2(M_1'm - M_2')}. \quad (41)$$

Where

$$\mathbb{Q} = \sqrt{4(-1+m)(M_2' - M_1'm)(M_2' - M_3' - M_1'm + M_2'm) + (3M_2' - M_3' + M_1'(-3+m)m)^2}$$

The moment estimators for φ_1, λ of $\mathrm{REIP}(\varphi_1, \lambda)$, φ_0, λ of ZIP (φ_0, λ), can be obtained by substituting $\varphi_0 = 0$, $\varphi_1 = 0$, respectively, in (36) and (37), then solving the simultaneous equations as before. The moment estimator for λ of $\mathrm{PD}(\lambda)$, can be obtained by substituting $\varphi_0 = \varphi_1 = 0$, in (36).

7. Application

Two applications using two real data sets are introduced to demonstrate the importance and flexibility of the proposed models. The performance of the models is assessed using goodness of fit test and different information criteria. The chi-squared (χ^2) test is applied for testing the goodness of fit of $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$ and its sub-models to the data sets. - log-likelihood, Akaike information criteria (AIC) and Bayesian information criteria (BIC) are used for comparing the models, Smaller values of– log-likelihood, AIC and BIC indicate better models.

7.1 The Number of Infected Cases with Covid 19 in Africa Data

An application using the number of infected cases with Covid 19 in 34 countries in the continent of Africa from 3/21/2020 to 12/10/2020 is illustrated. The data set is obtained from the site of World Health Organization (WHO). The observed frequency of the cases which are uninfected with Covid 19 are 162, thus the data set contains 162 zeros. and the observed frequency of one infected case with Covid 19 are 148.; i.e. the data set contains non- negligible number of zeros (left –endpoint) and one (right-endpoint).The bar chart and the normal Q-Q plot of the number of infected cases with Covid 19 in 34 countries in the continent of Africa from 3/21/2020 to 12/10/2020 is presented in Figures 2 and 3. It is noticed that the data contains inflation at two points 0 and 1.

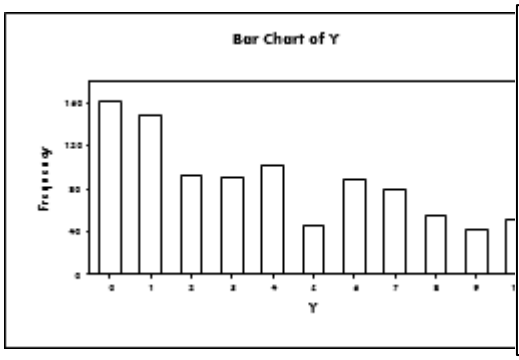

Figure 2: The Number of Infected Cases with Covid 19 in 34 Countries in the Continent of Africa from 3/21/2020 to 12/10/2020.

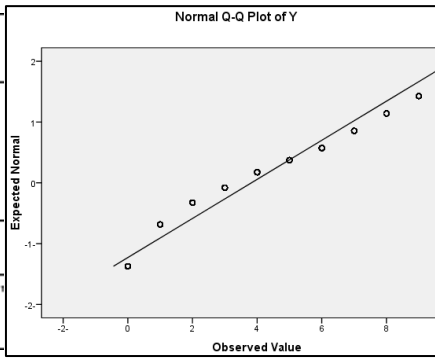

Figure 3: The Normal Q-Q Plot of the Number of Infected Cases with Covid 19 in 34 Countries in the Continent of Africa from 3/21/2020 to 12/10/2020

The observed and fitted frequency distributions based on the MLE and ME of the $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$ and its sub-models are presented in Table 1.

Table 1: Observed and Fitted Frequency Distributions of the Number of Infected Cases with Covid 19 in 34 Countries in the Continent of Africa from 3/21/2020 to 12/10/2020.

Observed Frequency		Count of infected cases										
		0	1	2	3	4	5	6	7	8	9	10
		162	148	93	91	102	46	89	79	54	42	52
MLE	$\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$	162	148	98	94	108	49	81	78	53	41	46
	$\mathrm{ZIP}(\varphi_0, \lambda)$	162	191	96	96	108	49	88	80	51	25	13
	$\mathrm{REIP}(\varphi_1, \lambda)$	24	144	90	88	118	140	110	119	72	36	16
	$\mathrm{PD}(\lambda)$	26	93	146	191	184	133	91	54	26	11	5
ME	$\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$	171	155	72	93	102	51	84	85	59	47	39
	$\mathrm{ZIP}(\varphi_0, \lambda)$	162	44	93	114	102	97	104	91	62	46	43
	$\mathrm{REIP}(\varphi_1, \lambda)$	5	430	23	47	71	85	86	74	56	37	44
	$\mathrm{PD}(\lambda)$	24	80	153	195	186	143	91	50	24	10	3

Point estimates with the corresponding standard errors and confidence intervals for the parameters of $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$ and its sub-models using the data set of the number of infected cases with Covid 19 in 34 countries in the continent of Africa from 3/21/2020 to 12/10/2020, are summarized in Table 2

Table 2: The Parameters Estimates and the Corresponding Standard Errors for the Models Using the Data Set of the Number of Infected Cases with Covid 19 in 34 Countries in the Continent of Africa from 3/21/2020 to 12/10/2020

		MLE			ME		
		Point	Std. Error	Interval	Point	Std. Error	Interval
$\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$	$\hat{\lambda}$	5.4223	0.0948	(5.2365, 5.6081)	5.6031	0.02421	(5.55565, 5.65055)
	$\hat{\varphi}_o$	0.1660	0.0118	0.1428, 0.1892)	0.1760	0.04308	(0.09156, 0.26044)
	$\hat{\varphi}_1$	0.1378	0.0118	0.1146, 0.1610)	0.1727	0.00167	(0.16943, 0.17597)
	$\widehat{E(Y)}$	3.9100			3.8226		
	$\widehat{V(Y)}$	9.0722			9.6616		
$\mathrm{ZIP}(\varphi_0,\lambda)$	$\hat{\lambda}$	4.6004	0.2857	(4.0405, 5.160)	5.3501	0.2754	(4.8104, 5.8898)
	$\hat{\varphi}_o$	0.1607	0.0122	(0.1367, 0.1847)	0.2855	0.0167	(0.2527, 0.3183)
	$\widehat{E(Y)}$	3.8613			3.8226		
	$\widehat{V(Y)}$	6.7154			9.6616		
$\mathrm{REIP}(\varphi_1,\lambda)$	$\hat{\lambda}$	4.3380	0.0793	(4.1826,4.4934)	6.0474	0.1030	(5.8456, 6.2492)
	$\hat{\varphi}_1$	0.1037	0.0126	(0.0789,0.1287)	0.4408	0.0173	(0.4069, 0.4747)
	$\widehat{E(Y)}$	3.9919			3.8225		
	$\widehat{V(Y)}$	4.9238			9.6616		
$\mathrm{PD}(\lambda)$	$\hat{\lambda}$	3.8225	0.06316	(3.6987,3.9463)	3.8225	0.0632	(3.6987,3.9463)
	$\widehat{E(Y)}$	3.8225			3.8225		
	$\widehat{V(Y)}$	3.8225			3.8225		

The estimated variance of the random variable $Y\sim\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$, $\widehat{V(Y)}$ reflect the variation of the data, which has more frequencies for some observations [namely zero counts and one counts] than other models.

Validation of the Models to the Data Set

The results of χ^2, - log-likelihood, AIC and BIC are summarized in Table 3.

Table 3: Validation of the Models to the Data Set of the Number of Infected Cases with Covid-19 in 34 Countries in the Continent of Africa from 3/21/2020 to 12/10/2020.

	Expected frequencies							
	MLE				ME			
	$\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$	$\mathrm{ZIP}(\varphi_0,\lambda)$	$\mathrm{REIP}(\varphi_1,\lambda)$	$\mathrm{PD}(\lambda)$	$\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$	$\mathrm{ZIP}(\varphi_0,\lambda)$	$\mathrm{REIP}(\varphi_1,\lambda)$	$\mathrm{PD}(\lambda)$
χ^2	3	146	973	1547	13	288	5885	1933
df	7	8	8	9	7	8	8	9
p-value	0.88500	<0.00001	<0.00001	<0.00001	0.072108	<0.00001	<0.00001	<0.00001
-Log-Likelihood	2298.87	2427.21	2649.07	2678.26	2307.65	2512.53	3090.59	2678.26
AIC	4603.75	4858.42	5302.14	5358.51	4621.30	5029.07	6185.18	5358.51
BIC	4618.34	4868.15	5311.88	5363.38	4635.89	5038.80	6194.91	5363.38

It is noticed from Table 3 that the data contains the $\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$ provides a better fit when compared with other models and can serve as an alternative model to the existing models.

7.2 The Number of Weekly Worked Days in Egypt Data.

An application using a sample of 18837 individuals in working ages (16-60 years) from 7526 family of the household income, expenditure and consumption survey (HIECS) carried out in Egypt at 2012- 2013 is conducted. The data set is obtained from the Central Agency for Public Mobilization and Statistics. Egypt, Arab Rep. 2012-2013. The

sample contains the number of days worked by 18837 individuals in the last week before the survey. The data set contains 8899 zeros and contains 4170 six; i.e. the data set contains non- negligible number of zeros (left –endpoint) and six (right-endpoint). The bar chart and the normal Q-Q plot of the number of days worked by 18837 individuals of 7526 family in Egypt in 2012 - 2013 is presented in Figures 4 and 5. It is noticed that the data contains inflation at two points 0 and 6.

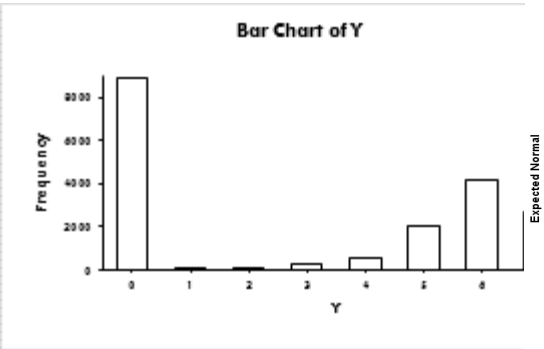

Figure 4: The Number of Days Worked by 18837 Individuals of 7526 Family in Egypt in 2012 – 2013

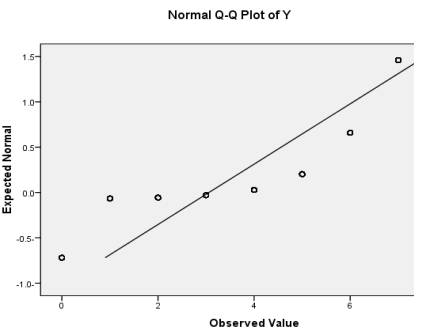

Figure 5: The normal Q-Q plot of the Number of Days Worked by 18837 Individuals of 7526 Family in Egypt in 2012 – 2013

The observed and fitted frequency distributions based on the MLE and ME of the $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$ and its sub-models are presented in Table 4.

Table 4: Observed and Fitted Frequency Distributions of the Number of Days Worked by 18837 Individuals of 7526 Family in Egypt in 2012 - 2013.

Observed Frequency		Count of days worked by an individual							
		0	1	2	3	4	5	6	7
		8899	54	89	307	562	2039	4170	2717
MLE	$\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$	8899	54	87	307	557	2039	4170	2722
	$\mathrm{ZIP}(\varphi_0, \lambda)$	8899	54	89	308	563	2037	4170	2717
	$\mathrm{REIP}(\varphi_1, \lambda)$	3269	54	89	307	562	1487	4134	177
	$\mathrm{PD}(\lambda)$	906	2658	4100	4215	3251	2005	1031	454
ME	$\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$	8896	52	87	308	563	2039	4173	2720
	$\mathrm{ZIP}(\varphi_0, \lambda)$	8899	54	89	307	562	2039	4170	2716
	$\mathrm{PD}(\lambda)$	906	2658	4100	4215	3251	2005	1031	454

Point estimates with the corresponding standard errors and confidence intervals for the parameters of $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$ and its sub-models using the data set of the number of days worked by 18837 individuals of 7526 family in Egypt in 2012 – 2013, are summarized in Table 5.

Table 5: The Parameters Estimates and the Corresponding Standard Errors for the Models Using the Data Set of the Number of Days Worked By 18837 Individuals of 7526 Family in Egypt in 2012 – 2013

		MLE			ME		
		Point	Std. Error	Interval	Point	Std. Error	Interval
$\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$	$\hat{\lambda}$	5.6545	0.0237	(5.6081, 5.7008)	5.0582	0.0487	(4.9627, 5.15365)
	$\hat{\varphi}_o$	0.4711	0.0032	0.4691, 0.4773)	0.4736	0.2399	(0.00340, 0.94380)
	$\hat{\varphi}_1$	0.1632	0.0025	0.1583, 0.1681)	0.4213	0.0142	(0.39347, 0.44913)
	$\widehat{E(Y)}$	3.0468			3.0597		
	$\widehat{V(Y)}$	10.3508			9.0274		
$\mathrm{ZIP}(\varphi_0,\lambda)$	$\hat{\lambda}$	5.7995	0.0243	(5.7519, 5.8470)	5.0101	0.0212	(4.9685, 5.0517)
	$\hat{\varphi}_o$	0.47082	0.0032	(0.4646, 0.4770)	0.3893	0.0032	(0.3831, 0.3955)
	$\widehat{E(Y)}$	3.0690			3.0597		
	$\widehat{V(Y)}$	11.4487			9.0274		
$\mathrm{REIP}(\varphi_1,\lambda)$	$\hat{\lambda}$	2.2237	0.0126	(2.1989, 2.2485)	1.0077	0.00547	(0.9970, 1.0184)
	$\hat{\varphi}_1$	0.2049	0.0031	(0.1989, 0.2109)	0.5110	0.00316	(0.5048, 0.5172)
	$\widehat{E(Y)}$	2.9975			3.5588		
	$\widehat{V(Y)}$	4.0914			6.7203		
$\mathrm{PD}(\lambda)$	$\hat{\lambda}$	3.0597	0.1275	(3.0347, 3.0846)	3.0597	0.1275	(3.0347, 3.0846)
	$\widehat{E(Y)}$	3.0597			3.0597		
	$\widehat{V(Y)}$	3.0597			3.0597		

The estimated variance of the random variable $Y\sim\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$ reflect the variation of the data, which has more frequencies for some observations [namely zero counts and six counts] than other models. The estimated variance of the random variable $Y\sim\mathrm{ZIP}(\varphi_0,\lambda)$ reflect the variation of the data which has more frequencies for zero counts than the other models.

Validation of the Models to the Data Set

The results of χ^2, - log-likelihood, AIC and BIC are summarized in Table 6.

Table 6: Validation of the Models to the Data Set of the Number of Days Worked by 18837 Individuals of 7526 Family in Egypt in 2012 – 2013.

	Expected frequencies						
	MLE				ME		
	$\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$	$\mathrm{ZIP}(\varphi_0,\lambda)$	$\mathrm{REIP}(\varphi_1,\lambda)$	$\mathrm{PD}(\lambda)$	$\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$	$\mathrm{ZIP}(\varphi_0,\lambda)$	$\mathrm{PD}(\lambda)$
$\chi 2$	0.08645	0.01021	46462	103678	0.18342	0.00295	103678
df	4	5	5	6	4	5	6
p-value	0.9991	0.9999	< 0.00001	< 0.00001	0.9960	1	< 0.00001
-Log-Likelihood	30198.60	32082.10	51561.40	55939.20	34715.20	32875.60	55939.20
AIC	60403.14	64168.23	103126.80	111880.50	69436.43	65755.17	111880.50
BIC	60426.67	64183.92	103142.50	111888.30	69459.96	65770.85	111888.30

It noticed from Table 6 that $\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$ provides a better fit when compare with other models and can serve as an alternative model to the existing models. The $\mathrm{ZIP}(\varphi_0,\lambda)$ provides a better fit when the counts have an upper bound.

8. Conclusions

- Estimation of the parameters of $\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$ and its sub-models by the method of maximum likelihood estimators and moment estimators is considered.
- The method of maximum likelihood estimators is shown to have better estimates on the two real data sets. The $\mathrm{EIP}(\varphi_0,\varphi_1,\lambda)$ is shown to have a better fitting for the frequencies of the real data sets than the existing models.

- The $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$ can be used for molding count data with both excessive zeros and excessive right-endpoint compared with other observations of the data that cannot be explained by models based on standard distributional assumptions as well as extra variation.
- The $\mathrm{PD}(\lambda)$ cannot deal with the proportion for incorporating extra zeros, φ_0 and extra right-endpoint, φ_1, while $\mathrm{ZIP}(\varphi_0, \lambda)$ and $\mathrm{REIP}(\varphi_1, \lambda)$ deal only with the proportion for incorporating extra zeros φ_0, or extra right-endpoint φ_1,respectively, which is greater than expected for the $\mathrm{PD}(\lambda)$, so, $\mathrm{ZIP}(\varphi_0, \lambda)$ and $\mathrm{REIP}(\varphi_1, \lambda)$ may give better estimates for molding count data with only an upper bound.

APPENDIX

APPENDIX A

The elements of the Hessian matrix of the ML estimators of the $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$, are obtained by taking the second derivatives of the natural logarithm of the likelihood function, $\ell(\underline{\vartheta})$ in (14) with respect to the unknown parameters, $\underline{\theta} = (\varphi_0, \varphi_1, \lambda)$,

The elements of the Hessian matrix of the ML estimators of the $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$, are obtained by taking the second derivatives of the natural logarithm of the likelihood function, $\ell(\underline{\vartheta})$ in (14) with respect to the unknown parameters, $\underline{\theta} = (\varphi_0, \varphi_1, \lambda)$,

The partial derivative of (15) with respect to φ_0 is given by:

$$J_{\varphi_o\varphi_o} = \frac{\partial^2 \ell(\underline{\theta})}{\partial {\varphi_0}^2} = \frac{-\left(1-e^{-\lambda}\right)^2}{(\varphi_o + \varphi_2 e^{-\lambda})^2} I_o - \frac{\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)^2}{\left(\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}\right)^2} I_1 - \frac{1}{{\varphi_2}^2} I_2. \qquad (\mathrm{A}.1)$$

The partial derivative of (16) with respect to φ_1 is given by:

$$J_{\varphi_1\varphi_1} = \frac{\partial^2 \ell(\underline{\theta})}{\partial {\varphi_1}^2} = -\frac{e^{-2\lambda}}{(\varphi_o + \varphi_2 e^{-\lambda})^2} I_o - \frac{\left(1-\frac{e^{-\lambda}\lambda^m}{m!}\right)^2}{\left(\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}\right)^2} I_1 - \frac{1}{{\varphi_2}^2} I_2. \qquad (\mathrm{A}.2)$$

The partial derivative of (17) with respect to λ is given by:

$$J_{\lambda\lambda} = \frac{\partial^2 \ell(\underline{\theta})}{\partial \lambda^2} = \frac{\varphi_o \varphi_2 e^{-\lambda}}{(\varphi_o + \varphi_2 e^{-\lambda})^2} I_o$$

$$= \frac{\varphi_2 \left(\varphi_1 e^{-\lambda} m \frac{\lambda^{m-2}}{(m-1)!} - \varphi_2 e^{-2\lambda} m \frac{\lambda^{2m-2}}{m!^2} - 2\varphi_1 e^{-\lambda} \frac{\lambda^{m-1}}{(m-1)!} - \varphi_1 e^{-\lambda} \frac{\lambda^{m-2}}{(m-1)!} + \varphi_1 e^{-\lambda} \frac{\lambda^m}{m!}\right)}{\left(\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}\right)^2} I_1 - \frac{n\bar{y}}{\lambda^2} \quad (A.3)$$

The partial derivative of (15) with respect to φ_1 is equal to the partial derivative of (16) with respect to φ_o and is given by

$$J_{\varphi_o\varphi_1} = J_{\varphi_1\varphi_o} = \frac{\partial^2 \ell(\underline{\theta})}{\partial \varphi_0 \partial \varphi_1} = \frac{\partial^2 \ell(\underline{\theta})}{\partial \varphi_1 \partial \varphi_o} = \frac{e^{-\lambda}\left(1-e^{-\lambda}\right)}{(\varphi_o + \varphi_2 e^{-\lambda})^2} I_o + \frac{\frac{e^{-\lambda}\lambda^m}{m!}\left(1-\frac{e^{-\lambda}\lambda^m}{m!}\right)}{\left(\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}\right)^2} I_1 - \frac{1}{{\varphi_2}^2} I_2. \qquad (\mathrm{A}.4)$$

The partial derivative of (15) with respect to λ is equal to the partial derivative of (17) with respect to φ_o and is given by

$$J_{\varphi_o\lambda} = J_{\lambda\varphi_0} = \frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_0\partial\lambda} = \frac{\partial^2 \ell(\underline{\theta})}{\partial\lambda\partial\varphi_0} = \frac{e^{-\lambda}(1-\varphi_1)}{(\varphi_o+\varphi_2 e^{-\lambda})^2} I_o - \frac{\varphi_1 e^{-\lambda}\left(\frac{\lambda^m}{m!} - \frac{\lambda^{m-1}}{(m-1)!}\right)}{\left(\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}\right)^2} I_1. \quad \text{(A.5)}$$

The partial derivative of (16) with respect to λ is equal the partial derivative of (17) with respect to φ_1 and is given by

$$J_{\varphi_1\lambda} = J_{\lambda\varphi_1} = \frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_1\partial\lambda} = \frac{\partial^2 \ell(\underline{\theta})}{\partial\lambda\partial\varphi_1} = \frac{\varphi_o e^{-\lambda}}{(\varphi_o+\varphi_2 e^{-\lambda})^2} I_o + \frac{(1-\varphi_o)e^{-\lambda}\left(\frac{\lambda^m}{m!} - \frac{\lambda^{m-1}}{(m-1)!}\right)}{\left(\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}\right)^2} I_1 \quad \text{(A.6)}$$

APPENDIX B

The elements of the Fisher information matrix of the ML estimators of the $\mathrm{EIP}(\varphi_0, \varphi_1, \lambda)$, are obtained by taking the negative expectation of the Hessian matrix as follows:

Note that:

$$\mathbb{E}(I_o) = P(y=0) = \varphi_o + \varphi_2 e^{-\lambda},$$

$$\mathbb{E}(I_1) = P(y=m) = \varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!},$$

and

$$\mathbb{E}(I_2) = \varphi_2 \frac{e^{-\lambda}\lambda^y}{y!}.$$

The expectation of (A.1) can be obtained as follows:

$$\mathrm{K}_{\varphi_o\varphi_o} = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_o^2}\right) = \frac{\left(1-e^{-\lambda}\right)^2}{\varphi_o+\varphi_2 e^{-\lambda}} + \frac{\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)^2}{\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}} + \frac{1}{\varphi_2}\sum_{y\neq 0,m}\left(\frac{e^{-\lambda}\lambda^y}{y!}\right). \quad \text{(B.1)}$$

The expectation of (A.2) can be obtained as follows:

$$\mathrm{K}_{\varphi_1\varphi_1} = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial{\varphi_1}^2}\right) \frac{e^{-2\lambda}}{\varphi_o+\varphi_2 e^{-\lambda}} + \frac{\left(1-\frac{e^{-\lambda}\lambda^m}{m!}\right)^2}{\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}} + \frac{1}{\varphi_2}\sum_{y\neq 0,m}\left(\frac{e^{-\lambda}\lambda^y}{y!}\right). \quad \text{(B.2)}$$

The expectation of (A.3) can be obtained as follows:

$$\mathrm{K}_{\lambda\lambda} = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial{\lambda}^2}\right) = \frac{-\varphi_o\varphi_2 e^{-\lambda}}{\varphi_o+\varphi_2 e^{-\lambda}}$$
$$- \frac{\varphi_2\left(\varphi_1 e^{-\lambda} m\frac{\lambda^{m-2}}{(m-1)!} - \varphi_2 e^{-2\lambda} m\frac{\lambda^{2m-2}}{m!^2} - 2\varphi_1 e^{-\lambda}\frac{\lambda^{m-1}}{(m-1)!} - \varphi_1 e^{-\lambda}\frac{\lambda^{m-2}}{(m-1)!} + \varphi_1 e^{-\lambda}\frac{\lambda^m}{m!}\right)}{\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}} + \frac{n\bar{y}}{\lambda^2} \quad \text{(B.3)}$$

The expectation of (A.4) can be obtained as follows:

$$\mathrm{K}_{\varphi_o\varphi_1} = \mathrm{K}_{\varphi_1\varphi_0} = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_0\partial\varphi_1}\right) = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_1\partial\varphi_o}\right)$$
$$= -\frac{e^{-\lambda}\left(1-e^{-\lambda}\right)}{\varphi_o+\varphi_2 e^{-\lambda}} - \frac{\frac{e^{-\lambda}\lambda^m}{m!}\left(1-\frac{e^{-\lambda}\lambda^m}{m!}\right)}{\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}} + \frac{1}{\varphi_2}\sum_{y\neq 0,m}\left(\frac{e^{-\lambda}\lambda^y}{y!}\right). \quad \text{(B.4)}$$

The expectation of (A.5) can be obtained as follows:

$$\mathrm{K}_{\varphi_o\lambda} = \mathrm{K}_{\lambda\varphi_0} = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_0\partial\lambda}\right) = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\lambda\partial\varphi_0}\right)$$
$$= \frac{-e^{-\lambda}(1-\varphi_1)}{\varphi_o + \varphi_2 e^{-\lambda}} + \frac{\varphi_1 e^{-\lambda}\left(\frac{\lambda^m}{m!} - \frac{\lambda^{m-1}}{(m-1)!}\right)}{\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}}. \qquad (B.5)$$

The expectation of (A.6) can be obtained as follows:

$$\mathrm{K}_{\varphi_1\lambda} = \mathrm{K}_{\lambda\varphi_1} = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_1\partial\lambda}\right) = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\lambda\partial\varphi_1}\right) = \frac{-\varphi_o e^{-\lambda}}{\varphi_o + \varphi_2 e^{-\lambda}} - \frac{(1-\varphi_o)e^{-\lambda}\left(\frac{\lambda^m}{m!} - \frac{\lambda^{m-1}}{(m-1)!}\right)}{\varphi_1 + \varphi_2 \frac{e^{-\lambda}\lambda^m}{m!}}. \qquad (B.6)$$

APPENDIX C

The elements of $\left|\mathrm{K}(\underline{\theta})\right|$ in (30), which is the determinant of the Fisher information matrix $\mathrm{K}(\underline{\theta})$, can be obtained as follows:

$$\left|K(\underline{\theta})\right| = \begin{vmatrix} K_{\varphi_o\varphi_o} & K_{\varphi_o\varphi_1} & K_{\varphi_o\lambda} \\ K_{\varphi_1\varphi_0} & K_{\varphi_1\varphi_1} & K_{\varphi_1\lambda} \\ K_{\lambda\varphi_0} & K_{\lambda\varphi_1} & K_{\lambda\lambda} \end{vmatrix}$$
$$= K_{\varphi_o\varphi_o}\left[K_{\varphi_1\varphi_1}K_{\lambda\lambda} - K_{\varphi_1\lambda}K_{\lambda\varphi_1}\right] - K_{\varphi_o\varphi_1}\left[K_{\varphi_1\varphi_0}K_{\lambda\lambda} - K_{\varphi_1\lambda}K_{\lambda\varphi_0}\right]$$
$$+K_{\varphi_o\lambda}\left[K_{\varphi_1\varphi_0}K_{\lambda\varphi_1} - K_{\varphi_1\varphi_1}K_{\lambda\varphi_0}\right], \qquad (C.1)$$

The elements of $adj\ \mathrm{K}(\underline{\theta})$ in (31), which is the adjoint of the Fisher information matrix $\mathrm{K}(\underline{\theta})$, can be obtained as follows:

$$\mathrm{cof}\ \mathrm{K}(\underline{\theta}) = \begin{pmatrix} \begin{vmatrix} K_{\varphi_1\varphi_1} & K_{\varphi_1\lambda} \\ K_{\lambda\varphi_1} & K_{\lambda\lambda} \end{vmatrix} & -\begin{vmatrix} K_{\varphi_1\varphi_0} & K_{\varphi_1\lambda} \\ K_{\lambda\varphi_0} & K_{\lambda\lambda} \end{vmatrix} & \begin{vmatrix} K_{\varphi_1\varphi_0} & K_{\varphi_1\varphi_1} \\ K_{\lambda\varphi_0} & K_{\lambda\varphi_1} \end{vmatrix} \\ -\begin{vmatrix} K_{\varphi_o\varphi_1} & K_{\varphi_o\lambda} \\ K_{\lambda\varphi_1} & K_{\lambda\lambda} \end{vmatrix} & \begin{vmatrix} K_{\varphi_o\varphi_o} & K_{\varphi_o\lambda} \\ K_{\lambda\varphi_0} & K_{\lambda\lambda} \end{vmatrix} & -\begin{vmatrix} K_{\varphi_o\varphi_o} & K_{\varphi_o\varphi_1} \\ K_{\lambda\varphi_0} & K_{\lambda\varphi_1} \end{vmatrix} \\ \begin{vmatrix} K_{\varphi_o\varphi_1} & K_{\varphi_o\lambda} \\ K_{\varphi_1\varphi_1} & K_{\varphi_1\lambda} \end{vmatrix} & -\begin{vmatrix} K_{\varphi_o\varphi_o} & K_{\varphi_o\lambda} \\ K_{\varphi_1\varphi_0} & K_{\varphi_1\lambda} \end{vmatrix} & \begin{vmatrix} K_{\varphi_o\varphi_o} & K_{\varphi_o\varphi_1} \\ K_{\varphi_1\varphi_0} & K_{\varphi_1\varphi_1} \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} K_{\varphi_1\varphi_1}K_{\lambda\lambda} - K_{\varphi_1\lambda}K_{\lambda\varphi_1} & -\left(K_{\varphi_1\varphi_0}K_{\lambda\lambda} - K_{\varphi_1\lambda}K_{\lambda\varphi_0}\right) & K_{\varphi_1\varphi_0}K_{\lambda\varphi_1} - K_{\varphi_1\varphi_1}K_{\lambda\varphi_0} \\ -\left(K_{\varphi_o\varphi_1}K_{\lambda\lambda} - K_{\varphi_o\lambda}K_{\lambda\varphi_1}\right) & K_{\varphi_o\varphi_o}K_{\lambda\lambda} - K_{\varphi_o\lambda}K_{\lambda\varphi_o} & -\left(K_{\varphi_o\varphi_o}K_{\lambda\varphi_1} - K_{\varphi_o\varphi_1}K_{\lambda\varphi_0}\right) \\ K_{\varphi_o\varphi_1}K_{\varphi_1\lambda} - K_{\varphi_o\lambda}K_{\varphi_1\varphi_1} & -\left(K_{\varphi_o\varphi_o}K_{\varphi_1\lambda} - K_{\varphi_o\lambda}K_{\varphi_1\varphi_0}\right) & K_{\varphi_o\varphi_o}K_{\varphi_1\varphi_1} - K_{\varphi_o\varphi_1}K_{\varphi_1\varphi_0} \end{pmatrix}. \qquad (C.2)$$

The transpose of (C.2) can be obtained as follows:

$$adj\ \mathrm{K}(\underline{\theta}) =$$
$$\begin{pmatrix} K_{\varphi_1\varphi_1}K_{\lambda\lambda} - K_{\varphi_1\lambda}K_{\lambda\varphi_1} & -\left(K_{\varphi_o\varphi_1}K_{\lambda\lambda} - K_{\varphi_o\lambda}K_{\lambda\varphi_1}\right) & K_{\varphi_o\varphi_1}K_{\varphi_1\lambda} - K_{\varphi_o\lambda}K_{\varphi_1\varphi_1} \\ -\left(K_{\varphi_1\varphi_0}K_{\lambda\lambda} - K_{\varphi_1\lambda}K_{\lambda\varphi_0}\right) & K_{\varphi_o\varphi_o}K_{\lambda\lambda} - K_{\varphi_o\lambda}K_{\lambda\varphi_o} & -\left(K_{\varphi_o\varphi_o}K_{\varphi_1\lambda} - K_{\varphi_o\lambda}K_{\varphi_1\varphi_0}\right) \\ K_{\varphi_1\varphi_0}K_{\lambda\varphi_1} - K_{\varphi_1\varphi_1}K_{\lambda\varphi_0} & -\left(K_{\varphi_o\varphi_o}K_{\lambda\varphi_1} - K_{\varphi_o\varphi_1}K_{\lambda\varphi_0}\right) & K_{\varphi_o\varphi_o}K_{\varphi_1\varphi_1} - K_{\varphi_o\varphi_1}K_{\varphi_1\varphi_0} \end{pmatrix}. \qquad (C.3)$$

By substituting (C.1) and (C.3) in (32), the elements of the variance-covariance matrix can be obtained as follows:

$$K^{\varphi_o\varphi_o} =$$
$$\frac{K_{\varphi_1\varphi_1}K_{\lambda\lambda} - K_{\varphi_1\lambda}K_{\lambda\varphi_1}}{K_{\varphi_o\varphi_o}\left[K_{\varphi_1\varphi_1}K_{\lambda\lambda} - K_{\varphi_1\lambda}K_{\lambda\varphi_1}\right] - K_{\varphi_o\varphi_1}\left[K_{\varphi_1\varphi_0}K_{\lambda\lambda} - K_{\varphi_1\lambda}K_{\lambda\varphi_0}\right] + K_{\varphi_o\lambda}\left[K_{\varphi_1\varphi_0}K_{\lambda\varphi_1} - K_{\varphi_1\varphi_1}K_{\lambda\varphi_0}\right]},$$
$$K^{\varphi_1\varphi_1} =$$

$$K^{\lambda\lambda} = \frac{K_{\varphi_o\varphi_o} K_{\lambda\lambda} - K_{\varphi_o\lambda} K_{\lambda\varphi_o}}{K_{\varphi_o\varphi_o}\left[K_{\varphi_1\varphi_1} K_{\lambda\lambda} - K_{\varphi_1\lambda} K_{\lambda\varphi_1}\right] - K_{\varphi_o\varphi_1}\left[K_{\varphi_1\varphi_0} K_{\lambda\lambda} - K_{\varphi_1\lambda} K_{\lambda\varphi_0}\right] + K_{\varphi_o\lambda}\left[K_{\varphi_1\varphi_0} K_{\lambda\varphi_1} - K_{\varphi_1\varphi_1} K_{\lambda\varphi_0}\right]},$$

$$K^{\varphi_o\varphi_1} = K^{\varphi_1\varphi_0} = \frac{K_{\varphi_o\varphi_o} K_{\varphi_1\varphi_1} - K_{\varphi_o\varphi_1} K_{\varphi_1\varphi_0}}{K_{\varphi_o\varphi_o}\left[K_{\varphi_1\varphi_1} K_{\lambda\lambda} - K_{\varphi_1\lambda} K_{\lambda\varphi_1}\right] - K_{\varphi_o\varphi_1}\left[K_{\varphi_1\varphi_0} K_{\lambda\lambda} - K_{\varphi_1\lambda} K_{\lambda\varphi_0}\right] + K_{\varphi_o\lambda}\left[K_{\varphi_1\varphi_0} K_{\lambda\varphi_1} - K_{\varphi_1\varphi_1} K_{\lambda\varphi_0}\right]},$$

$$K^{\varphi_o\lambda} = K^{\lambda\varphi_o} = \frac{-\left(K_{\varphi_o\varphi_1} K_{\lambda\lambda} - K_{\varphi_o\lambda} K_{\lambda\varphi_1}\right)}{K_{\varphi_o\varphi_o}\left[K_{\varphi_1\varphi_1} K_{\lambda\lambda} - K_{\varphi_1\lambda} K_{\lambda\varphi_1}\right] - K_{\varphi_o\varphi_1}\left[K_{\varphi_1\varphi_0} K_{\lambda\lambda} - K_{\varphi_1\lambda} K_{\lambda\varphi_0}\right] + K_{\varphi_o\lambda}\left[K_{\varphi_1\varphi_0} K_{\lambda\varphi_1} - K_{\varphi_1\varphi_1} K_{\lambda\varphi_0}\right]},$$

$$K^{\varphi_1\lambda} = K^{\lambda\varphi_1} = \frac{K_{\varphi_o\varphi_1} K_{\varphi_1\lambda} - K_{\varphi_o\lambda} K_{\varphi_1\varphi_1}}{K_{\varphi_o\varphi_o}\left[K_{\varphi_1\varphi_1} K_{\lambda\lambda} - K_{\varphi_1\lambda} K_{\lambda\varphi_1}\right] - K_{\varphi_o\varphi_1}\left[K_{\varphi_1\varphi_0} K_{\lambda\lambda} - K_{\varphi_1\lambda} K_{\lambda\varphi_0}\right] + K_{\varphi_o\lambda}\left[K_{\varphi_1\varphi_0} K_{\lambda\varphi_1} - K_{\varphi_1\varphi_1} K_{\lambda\varphi_0}\right]},$$

$$K^{\varphi_1\varphi_1} = \frac{-\left(K_{\varphi_o\varphi_o} K_{\varphi_1\lambda} - K_{\varphi_o\lambda} K_{\varphi_1\varphi_0}\right)}{K_{\varphi_o\varphi_o}\left[K_{\varphi_1\varphi_1} K_{\lambda\lambda} - K_{\varphi_1\lambda} K_{\lambda\varphi_1}\right] - K_{\varphi_o\varphi_1}\left[K_{\varphi_1\varphi_0} K_{\lambda\lambda} - K_{\varphi_1\lambda} K_{\lambda\varphi_0}\right] + K_{\varphi_o\lambda}\left[K_{\varphi_1\varphi_0} K_{\lambda\varphi_1} - K_{\varphi_1\varphi_1} K_{\lambda\varphi_0}\right]},$$

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