The Topp -Leone Compound Pareto Type II-gamma $(\boldsymbol{\theta}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{c})$ Distribution: Properties and Estimation

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تاريخ استلام البحث: 2022/11/27

تاريخ قبول البحوث: 2023/01/24

نشر البحث في العدد الثامن عشر: مارس / اذار 2023

رمز التصنيف ديوي / النسخة الالكترونية (Online): 2522-64X/519.5

رمز التصنيف ديوي / النسخة الورقية (Print): 2519-948X/519.5

The Topp -Leone Compound Pareto Type II-gamma $(\boldsymbol{\theta}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{c})$ Distribution: Properties and Estimation

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Abstract

This paper is concerned with the Topp-Leone compound Pareto Type II-gamma distribution [TL-CPIIG (θ, α, β, c)] with its mathematical forms of reliability function (rf), hazard rate function $\mathrm{h}(\mathrm{x})$, reversed hazard function (rhf), cumulative hazard function $\mathrm{H}_{\mathrm{TL}}(\mathrm{x})$, the approximate mean and variance, mode, quantile, median and order statistics are obtained. The maximum likelihood method is used to estimate the parameters numerically. Simulation is used to represent the performance of the suggested distribution; empirical results of new model are applied by modeling two real data sets of this study.

Keywords: *Topp-Leone compound Pareto Type II-gamma distribution; reliability function; hazard rate function; reversed hazard function; cumulative hazard function; the approximate mean and variance; mode; quartile; median; order statistics; maximum likelihood estimation; Monte Carlo simulations.*

1. Introduction

This paper deals with a very important subject in nowadays, with numerous compounds or contagious phenomena, which can't be explained as simple events. Many authors studied the Topp-Leone (TL) distribution and its applications. The (TL) distribution is one of the continuous distributions that is attractive as a generator. This distribution was proposed by Topp and Leone (1955). It provides closed forms of the cumulative distribution function (cdf) and probability density function (pdf). According to the number of parameters, the estimation part for the TL distribution is not complicated. However, the TL distribution had not received much attention until Nadarajah and Kotz (2003) discovered it. In addition, they studied some properties of the TL distribution and provided its moments, central moments and characteristic function.

Recently, applying new generators for continuous distributions became more interesting. This methodology can improve the goodness of fit and determine tail properties and let the new distribution more flexible to model real data. These features have been established by the results of many generators such as beta distribution, generalized Kumaraswamy distribution, generalized beta distribution and the exponentiated family of distributions [See Eugene *et al*. (2002), Jones (2004), Cordeiro and de Castro (2011), Alexander *et al*. (2012), AL-Hussaini and Ahsanullah (2015)].

On the other hand, compound distributions gained their importance from the fact that factors have compound effects, as in medical, biological and social experiments which make it very difficult to represent these phenomena by simple distributions. These generalized distributions give more flexibility by adding new parameter to the baseline model and they useful in obtaining general results that could be applied to special cases to obtain new results.
For the conditional distribution X of the Pareto distribution, given the parameters a and c, the pdf is given by:

$$f(x \mid a, c) = ac^{a}(x+c)^{-(a+1)}, \qquad x > 0, \quad (a, c > 0) \qquad (1)$$

where a has gamma distribution, with parameters α and β, and pdf is given by

$$f(a) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} a^{\alpha-1} e^{-\beta a}, \qquad a > 0, \quad (\alpha, \beta > 0) \qquad (2)$$

The cdf of the random variable X is given by

$$F(x) = 1 - \beta^{\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-\alpha}, \quad x \geq 0, \quad (\alpha, \beta, c \geq 0). \qquad (3)$$

This is the cdf of the $\mathrm{CPIIG}(\alpha, \beta, c)$ distribution.

The pdf of the compound distribution is given by

$$f(x) = \alpha\beta^{\alpha}(x+c)^{-1}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(\alpha+1)}, \quad x \geq 0, \quad (\alpha, \beta, c \geq 0) \qquad (4)$$

[See Youssef (2006)].

This paper organized as follows: In Section 2 Topp-Leone compound Pareto Type II-gamma distribution, reliability function (rf), hazard rate function $h(x)$, reversed hazard function (rhf) and cumulative hazard function $H_{TL}(x)$ are obtained. The approximate mean and variance, also some properties of the distribution are contained in Section 3. Some special cases of the distribution are obtained in Section 4. In Section 5 some related distributions are obtained. Estimation of the parameters of Topp-Leone compound Pareto Type II-gamma (θ, α, β, c) distribution using maximum likelihood method is obtained in Section 6. Finally, numerical study is introduced and concluding remarks are presented to illustrate the theoretical results derived for ML estimation in Section 7.

2. The Topp-Leone compound Pareto Type II-gamma (θ, α, β, c) distribution.

The Topp-Leone compound Pareto Type II-gamma distribution with parameters θ, α, β and c, which will be refer as *TL-CPIIG (ϑ, α, β, c)*, can be derived.

If a random variable X is distributed as the (TL) and bounded on [0, 1]. Let X be a continuous random variable with (cdf) G(x). The (TLG) distribution has (cdf) written by

$$F_{TLG}(x) = \big(G(x)\big)^{\theta}\,(2 - G(x))^{\theta}, \qquad 0 < x < \infty, \qquad \theta > 0 \qquad (5)$$

By differentiating, the corresponding (pdf) is

$$f_{TLG}(x) = 2\theta\, g(x)(1 - G(x))\big(G(x)\big)^{\theta-1}(2 - G(x))^{\theta-1}, \; 0 < x < \infty, \quad \theta > 0 \qquad (6)$$

Where $g(x) = \frac{dG(x)}{dx}$ and θ is a shape parameter.

By combining (3), (4) into (5), (6) hence the pdf and cdf of the TL-CPIIG $(\theta, \alpha, \beta, c)$ are given, respectively, as follows:

$$F_{TL-CP}(x) = \left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}, \quad x \geq 0, \qquad (\theta, \alpha, \beta, c \geq 0) \qquad (7)$$

$$f_{TL-CP}(x) = 2\theta\alpha\beta^{2\alpha}(x+c)^{-1}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)}\left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1}, \; x \geq 0,$$
$$(\theta, \alpha, \beta, c \geq 0), \quad (8)$$

Where (θ, α, β) are shape parameters and c is a scale parameter.

For real value of θ, using following series representation of Prudnikov *et al* (1986):

$$(1+x)^{\theta} = \sum_{j=0}^{\infty} \frac{(1)^j \Gamma(\theta+1)}{j!\,\Gamma(\theta+1-j)} x^j = \sum_{j=0}^{\infty} (1)^j \binom{\theta}{j} x^j, \qquad \text{[See Abbas } et\ al \text{ (2017)]}.$$

The cdf of TL-CPIIG (θ, α, β, c) distribution given in (7) is expressed as infinite sum given as follows

$$F_{TL-CP}(x) = \left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\theta+1)}{j!\,\Gamma(\theta+1-j)}\, \beta^{2\alpha j}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha j} \qquad (9)$$

A density function of (8) TL-CPIIG $(\theta, \alpha, \beta, c)$ distribution can be written as follows:

$$f_{TL-CP}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j 2\alpha\theta\Gamma(\theta)}{j!\,\Gamma(\theta-j)}\, \beta^{2\alpha(j+1)}(x+c)^{-1}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha(j+1)-1} \qquad (10)$$

The pdf of TL-CPIIG density function for (θ, α, β, c) can represent in figure 1.

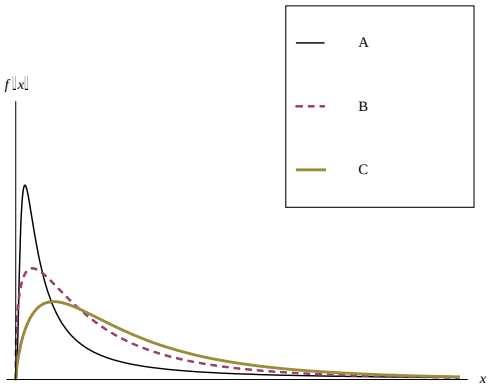

Figure 1: Probability density function for the TL-CPIIG distribution for different parameter values

(A) $(\theta = 5,\ \alpha = 0.44,\ \beta = 0.11,\ c = 0.43)$
(B) $(\theta = 1.37,\ \alpha = 4.5,\ \beta = 1.45,\ c = 4.43)$
(C) $(\theta = 1.72,\ \alpha = 9,\ \beta = 2.4,\ c = 6.5)$

From Figure 1, it is noticed that the pdf curve is right skewed in (A, B). While, the pdf curve is oblate flat kurtosis in (C).

The (rf) of the TL-CPIIG $(\theta, \alpha, \beta, c)$ is given by:

$$R_{TL-CP}(x) = 1 - F_{TL-CP}(x) = 1 - \left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}, x \geq 0, (\theta, \alpha, \beta, c \geq 0) \quad (11)$$

The (hrf) of the TL-CPIIG $(\theta, \alpha, \beta, c)$ is given by

$$h_{TL-CP}(x) = \frac{f_{TL-CP}(x)}{1 - F_{TL-CP}(x)} = \frac{2\theta\alpha\beta^{2\alpha}(x+c)^{-1}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)}\left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1}}{1 - \left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}},$$

$$x \geq 0, \qquad (\theta, \alpha, \beta, c \geq 0) \qquad (12)$$

The $(\boldsymbol{hrf})$ of the TL-CPIIG for different parameter values in presented in figure2

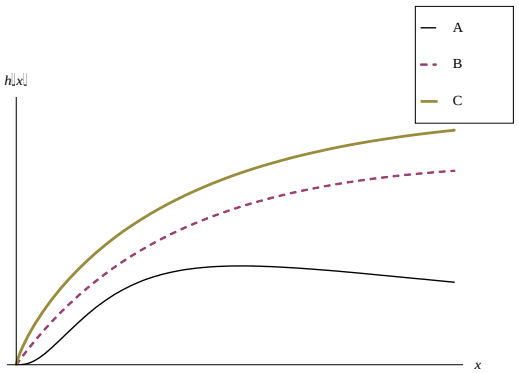

Figure 2: The hazard of the TL-CPIIG distribution for different parameter values

(A) $(\theta = 4,\ \alpha = 0.54,\ \beta = 0.8,\ c = 0.5)$
(B) $(\theta = 2,\ \alpha = 3,\ \beta = 1.6,\ c = 5)$
(C) $(\theta = 1.8,\ \alpha = 7.5,\ \beta = 2.6,\ c = 8)$

From Figure 2, it is notice that the (hrf) curve is monotonic increasing**.**
The $(rhrf)$ of RT TL-CPIIG $(\theta, \alpha, \beta, c)$ is given by:

- $$\mathrm{rh}_{\mathrm{TL-CP}}(x) = \frac{f_{\mathrm{TL-CP}}(x)}{F_{\mathrm{TL-CP}}(x)} = \frac{2\theta\alpha\beta^{2\alpha}(x+c)^{-1}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)}}{\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)}, x \geq 0, \quad (\theta, \alpha, \beta, c \geq 0) \quad (13)$$

The $(H_{TL}(x))$ of the TL-CPIIG $(\theta, \alpha, \beta, c)$ is

- $$H_{\mathrm{TL-CP}}(x) = -\ln\left(1 - F_{\mathrm{TL-CP}}(x)\right) = \ln\left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}, \quad x \geq 0, \quad (\theta, \alpha, \beta, c \geq 0), \quad (14)$$

3. Some Properties of the Distribution

In this section some important and useful statistical characteristics of the proposed distribution is derived.

- **The approximate Mean and Variance:**

The r th moment about zero of the random variable x can be obtained as follows:

$$\mu'_r = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

$$= 2\theta\alpha\beta^{2\alpha} \int_0^{\infty} x^r (x+c)^{-1} \left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)} \left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1} dx \quad (15)$$

It is difficult to find it in a closed form but the approximate mean and variance of the TL-CPIIG (θ, α, β, c) distribution can be obtained as follows:

If y is a random variable distributed as an exponential distribution with parameter θ [$Y \sim Exp(\theta)$] with $\mu = E(y) = \left(\frac{1}{\theta}\right)$ and $\sigma^2 = Var(y) = \frac{1}{\theta^2}$, then the variable $T = g(y) = c\left(e^{-\beta\left(1-(1-e^{-y})^{-\frac{1}{2\alpha}}\right)} - 1\right) \sim TL - CPIIG\ (\theta, \alpha, \beta, c)$. This relation will be used to find the approximate mean and variance of $TL - CPIIG\ (\theta, \alpha, \beta, c)$. The approximate mean and variance of g(y), based on the method of statistical differentials [See El-Sayad (1993)], are given by:

$$E(g(Y)) \cong g(\mu) + \frac{1}{2}\sigma^2 g''(\mu) \qquad (15a)$$

and

$$Var(g(Y)) \cong \sigma^2 \left(g'(\mu)\right)^2 \qquad (15b)$$

The function of μ is given by:

$$g(\mu) = c\left(e^{-\beta\left(1-(1-e^{-\mu})^{-\frac{1}{2\alpha}}\right)} - 1\right) \qquad (15c)$$

The first and second derivatives are given by:

$$g'(\mu) = -\frac{c\beta}{2\alpha}(1-e^{-\mu})^{-\left(\frac{1}{2\alpha}+1\right)} e^{-\left(\mu+\beta\left(1-(1-e^{-\mu})^{-\frac{1}{2\alpha}}\right)\right)}, \qquad (15d)$$

$$g''(\mu) = \frac{c\beta}{2\alpha}(1-e^{-\mu})^{-\left(\frac{1}{2\alpha}+2\right)} e^{-\left(2\mu+\beta\left(1-(1-e^{-\mu})^{-\frac{1}{2\alpha}}\right)\right)} \left[e^{\mu} + \frac{1}{2\alpha}\left(1-\beta(1-e^{-\mu})^{-\frac{1}{2\alpha}}\right)\right] \qquad (15e)$$

By substituting $(11c)$ and $(11e)$ in $(11a)$ we obtain

$$E(g(Y)) \cong c\left(e^{-\beta\left(1-(1-e^{-\mu})^{-\frac{1}{2\alpha}}\right)} - 1\right) + \frac{c\beta}{4\alpha\theta^2}(1-e^{-\mu})^{-\left(\frac{1}{2\alpha}+2\right)} e^{-\left(2\mu+\beta\left(1-(1-e^{-\mu})^{-\frac{1}{2\alpha}}\right)\right)} \times \left[e^{\mu} + \frac{1}{2\alpha}\left(1-\beta(1-e^{-\mu})^{-\frac{1}{2\alpha}}\right)\right] \qquad (16)$$

and by substituting $(15d)$ in $(15b)$ we obtain

$$Var(g(Y)) \cong \frac{c^2\beta^2}{4\alpha^2\theta^2}(1-e^{-\mu})^{-2\left(\frac{1}{2\alpha}+1\right)} e^{-2\left(\mu+\beta\left(1-(1-e^{-\mu})^{-\frac{1}{2\alpha}}\right)\right)} \qquad (17)$$

- **The mode:**

The mode of the TL-CPIIG (θ, α, β, c) distribution (8) can be obtained by differentiating the pdf with respect to x and equating the resulting equation to zero as $\grave{f}_{TL-CP}(x) = 0$ as follows:

$$\grave{f}_{TL-CP}(x) = 2\theta\alpha\beta^{2\alpha}\Bigg[2\alpha\beta^{2\alpha}(\theta - 1)(x+c)^{-2}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(4\alpha-2)}\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-2} - (2\alpha+1)(x+c)^{-2}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha-2)}\left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1} - (x+c)^{-2}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha-1)}\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1}\Bigg] = 0$$

$$\grave{f}_{TL-CP}(x) = 2\theta\alpha\beta^{2\alpha}(x+c)^{-2}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha-1)}\left(-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1}$$

$$\grave{F}_{TL-CP}(x) = 2\alpha\beta^{2\alpha}(\theta-1)\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha-1)}\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{-1} -(2\alpha+1)\left(\beta - \ln\left(\frac{c}{x+c}\right)\right) - 1 = 0 \qquad (18)$$

It is difficult to find it in a closed form.

- **Quantile:**

The quantile of the TL-CPIIG (θ, α, β, c) distribution can be obtained by putting $F_{TL-CP}(x) = q$ to obtain the quantile as follows:

$$F_{TL-CP}(x) = \int_0^x f_{TL-CP}(x)\,dx = \left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta} = q$$

Which yields:

$$x_q = c\left(e^{-\beta\left(1-\left(1-q^{\frac{1}{\theta}}\right)^{-\frac{1}{2\alpha}}\right)} - 1\right), \qquad (19)$$

The median is obtained at $q = \frac{1}{2}$ as follows:

$$x_{\frac{1}{2}} = c\left(e^{-\beta\left(1-\left(1-2^{-\frac{1}{\theta}}\right)^{-\frac{1}{2\alpha}}\right)} - 1\right), \qquad (20)$$

- **Order statistics**

Let $X_1, X_2, \ldots, X_n$ be i.i.d random variables from the TL-CPIIG (θ, α, β, c) distribution. Let Y_i denotes the i^{th} order statistic of $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$.

It is well known that, from the pdf of the i^{th} order statistic y_i of a random sample of size n drawn from a population with pdf f(y) and cdf F(y) is given by

$$f_{Y_{(i)}}(y) = i\binom{n}{i} f_Y(y)[F_Y(y)]^{i-1}[1-F_Y(y)]^{n-i}, \quad (21)$$

By substituting f(y), F(y), given by (8) and (7) after replacing y by y_i we obtain

$$f_{Y_{(i)}}(y) = i\binom{n}{i} 2\theta\alpha\beta^{2\alpha}(y+c)^{-1}\left(\beta - \ln\left(\frac{c}{y+c}\right)\right)^{-(2\alpha+1)}\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{y+c}\right)\right)^{-2\alpha}\right)^{\theta-1}$$

$$\times\left[\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{y+c}\right)\right)^{-2\alpha}\right)^{\theta}\right]^{i-1}\left[1-\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{y+c}\right)\right)^{-2\alpha}\right)^{\theta}\right]^{n-i}, 0<y<\infty$$

$$f_{Y_{(i)}}(y) = i\binom{n}{i} 2\theta\alpha\beta^{2\alpha}(y+c)^{-1}\left(\beta - \ln\left(\frac{c}{y+c}\right)\right)^{-(2\alpha+1)}\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{y+c}\right)\right)^{-2\alpha}\right)^{\theta i-1}$$

$$\times\left[1-\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{y+c}\right)\right)^{-2\alpha}\right)^{\theta}\right]^{n-i}, \quad 0<y<\infty \quad (22)$$

Special cases:

(i) If in (22), i =n, we obtain the pdf of the last order statistic, $Y_{(n)} = \max_{1\le a\le n}\{Y_{(j)}\}$, which is given by:

$$f_{Y_{(n)}}(y) = 2n\theta\alpha\beta^{2\alpha}(y+c)^{-1}\left(\beta - \ln\left(\frac{c}{y+c}\right)\right)^{-(2\alpha+1)}\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{y+c}\right)\right)^{-2\alpha}\right)^{\theta n-1} \quad 0<y<\infty, \quad (23)$$

If i = 1 in (22), we obtain the pdf of the first order statistic, $Y_{(1)} = \min_{1\le a\le n}\{Y_{(j)}\}$, which is given by:

$$f_{Y_{(1)}}(y) = 2n\theta\alpha\beta^{2\alpha}(y+c)^{-1}\left(\beta - \ln\left(\frac{c}{y+c}\right)\right)^{-(2\alpha+1)}\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{y+c}\right)\right)^{-2\alpha}\right)^{\theta-1}$$

$$\times\left[1-\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{y+c}\right)\right)^{-2\alpha}\right)^{\theta}\right]^{n-1}, 0<y<\infty, \quad (24)$$

(ii) The joint density of $Y_{(i)}$ and $Y_{(j)}$ can be obtained by substituting (7) and (8) in the joint density function of $Y_{(i)}$ and $Y_{(j)}$ $(1 \le i < j \le n)$ is given by:

$$f_{Y_{(i)},Y_{(j)}}(x,y) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} f(x)\, f(y)[F(x)]^{i-1}\ [1-F(y)]^{n-j}[F(y)-F(x)]^{j-i-1}\ ,$$

$-\infty < x < y < \infty$ (25)

[See Arnold, BalKrishnan and Nagaraja (1992)].

$$f_{Y_{(i)},Y_{(j)}}(x,y) = \frac{4n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\theta^2\alpha^2\beta^{4\alpha}(x+c)^{-1}(y+c)^{-1}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)}$$

$$\times\left(\beta-\ln\left(\frac{c}{y+c}\right)\right)^{-(2\alpha+1)}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{y+c}\right)\right)^{-2\alpha}\right)^{\theta-1}$$

$$\times\left[\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}\right]^{i-1}\left[1-\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{y+c}\right)\right)^{-2\alpha}\right)^{\theta}\right]^{n-j}$$

$$\times\left[\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{y+c}\right)\right)^{-2\alpha}\right)^{\theta}-\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}\right]^{j-i-1}$$

$$, \quad 0<x<y<\infty$$

$$f_{Y_{(i)},Y_{(j)}}(x,y)=\frac{4n!}{(i-1)!(j-i-1)!(n-j)!}\theta^2\alpha^2\beta^{4\alpha}(x+c)^{-1}(y+c)^{-1}\big(A(x)\big)^{-(2\alpha+1)}\big(A(y)\big)^{-(2\alpha+1)}$$

$$\times\left(1-\beta^{2\alpha}\big(A(x)\big)^{-2\alpha}\right)^{\theta-1}\left(1-\beta^{2\alpha}\big(A(y)\big)^{-2\alpha}\right)^{\theta-1}\left[\left(1-\beta^{2\alpha}\big(A(x)\big)^{-2\alpha}\right)^{\theta}\right]^{i-1}$$

$$\times\left[1-\left(1-\beta^{2\alpha}\big(A(y)\big)^{-2\alpha}\right)^{\theta}\right]^{n-j}\left[\left(1-\beta^{2\alpha}\big(A(y)\big)^{-2\alpha}\right)^{\theta}-\left(1-\beta^{2\alpha}\big(A(x)\big)^{-2\alpha}\right)^{\theta}\right]^{j-i-1},$$

$$0<x<y<\infty, \qquad (26)$$

where

$$A(x)=\left(\beta-\ln\left(\frac{c}{x+c}\right)\right) \quad \text{and} \quad A(y)=\left(\beta-\ln\left(\frac{c}{y+c}\right)\right) \qquad (27)$$

Then, when i = 1, j = n, we obtain, the joint pdf of $Y_{(1)}, Y_{(n)}$

$$f_{Y_{(1)},Y_{(n)}}(x,y)=n(n-1)\,f(x)\,f(y)[F(y)-F(x)]^{n-2}, \qquad 0<x<y<\infty$$

$$=4n(n-1)\theta^2\alpha^2\beta^{4\alpha}(x+c)^{-1}(y+c)^{-1}\big(A(x)\big)^{-(2\alpha+1)}\big(A(y)\big)^{-(2\alpha+1)}$$

$$\times\left(1-\beta^{2\alpha}\big(A(x)\big)^{-2\alpha}\right)^{\theta-1}\left(1-\beta^{2\alpha}\big(A(y)\big)^{-2\alpha}\right)^{\theta-1}$$

$$\times\left[\left(1-\beta^{2\alpha}\big(A(y)\big)^{-2\alpha}\right)^{\theta}-\left(1-\beta^{2\alpha}\big(A(x)\big)^{-2\alpha}\right)^{\theta}\right]^{n-2}, 0<x<y<\infty. \quad (28)$$

4. Special cases of Topp-Leone compound Pareto Type II-gamma (θ, α, β, c) distribution

Some special cases of the distribution can be obtained by putting one or more of the parameters equal to specific values. The following are some of the special cases of the distribution.

1- For $\theta=1$, we get the TL-CPIIG (θ, α, β, c) distribution with three parameters. Then, the pdf can be obtained as:

$$f_{TL-CP}(x)=2\alpha\beta^{2\alpha}(x+c)^{-1}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)}, \quad x\geq 0, \quad (\alpha,\beta,c\geq 0). \qquad (29)$$

2- For $\alpha = 1$, we get,

$$f_{TL-CP}(x) = 2\theta\beta^2(x+c)^{-1}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-3}\left(1-\beta^2\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2}\right)^{\theta-1},$$

$x \geq 0, \quad (\theta, \beta, c \geq 0). \quad (30)$

3- For $\beta = 1$, we get,

$$f_{TL-CP}(x) = 2\theta\alpha(x+c)^{-1}\left(1 - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)}\left(1-\left(1 - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1},$$

$x \geq 0, \quad (\theta, \alpha, c \geq 0), \quad (31)$

4- For $c = 1$, we get,

$$f_{TL-CP}(x) = 2\theta\alpha\beta^{2\alpha}(x+1)^{-1}\left(\beta - \ln\left(\frac{1}{x+1}\right)\right)^{-(2\alpha+1)}\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{1}{x+1}\right)\right)^{-2\alpha}\right)^{\theta-1},$$

$x \geq 0, \quad (\theta, \alpha, \beta \geq 0), \quad (32)$

5- For $\theta = 1, \alpha = 1$

$$f_{TL-CP}(x) = 2\beta^2(x+c)^{-1}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-3}, \quad x \geq 0, \quad (\beta, c \geq 0), \quad (33)$$

6- For $\theta = 1, \beta = 1$

$$f_{TL-CP}(x) = 2\alpha(x+c)^{-1}\left(1 - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)}, \quad x \geq 0, \quad (\alpha, c \geq 0), \quad (34)$$

7- For $\theta = 1, c = 1$

$$f_{TL-CP}(x) = 2\alpha\beta^{2\alpha}(x+1)^{-1}\left(\beta - \ln\left(\frac{1}{x+1}\right)\right)^{-(2\alpha+1)}, \quad x \geq 0, \quad (\alpha, \beta \geq 0), \quad (35)$$

8- For $\alpha = 1, \beta = 1$

$$f_{TL-CP}(x) = 2\theta(x+c)^{-1}\left(1 - \ln\left(\frac{c}{x+c}\right)\right)^{-3}\left(1-\left(1 - \ln\left(\frac{c}{x+c}\right)\right)^{-2}\right)^{\theta-1}, \quad x \geq 0, \quad (\theta, c \geq 0), \quad (36)$$

9- For $\alpha = 1, c = 1$

$$f_{TL-CP}(x) = 2\theta\beta^2(x+1)^{-1}\left(\beta - \ln\left(\frac{1}{x+1}\right)\right)^{-3}\left(1-\beta^2\left(\beta - \ln\left(\frac{1}{x+1}\right)\right)^{-2}\right)^{\theta-1},$$

$x \geq 0, \quad (\theta, \beta \geq 0), \quad (37)$

10- For $\beta = 1, c = 1$

$$f_{TL-CP}(x) = 2\theta\alpha(x+1)^{-1}\left(1 - \ln\left(\frac{1}{x+1}\right)\right)^{-(2\alpha+1)}\left(1-\left(1 - \ln\left(\frac{1}{x+1}\right)\right)^{-2\alpha}\right)^{\theta-1},$$

$x \geq 0, \quad (\theta, \alpha \geq 0), \quad (38)$

11- For $\theta = 1, \alpha = 1, \beta = 1$

$$f_{TL-CP}(x) = 2(x+c)^{-1}\left(1 - \ln\left(\frac{c}{x+c}\right)\right)^{-3}, \quad x \geq 0, \quad (c \geq 0), \quad (39)$$

12- For $\theta = 1, \alpha = 1, c = 1$

$$f_{TL-CP}(x) = 2\beta^2(x+1)^{-1}\left(\beta - \ln\left(\frac{1}{x+1}\right)\right)^{-3}, \quad x \geq 0, \quad (\beta \geq 0), \quad (40)$$

13- For $\alpha = 1, \beta = 1, c = 1$

$$f_{TL-CP}(x) = 2\theta(x+1)^{-1}\left(1-\ln\left(\frac{1}{x+1}\right)\right)^{-3}\left(1-\left(1-\ln\left(\frac{1}{x+1}\right)\right)^{-2}\right)^{\theta-1}, x \geq 0, \quad (\theta \geq 0). \quad (41)$$

14- For $\theta = 1, \beta = 1, c = 1$

$$f_{TL-CP}(x) = 2\alpha(x+1)^{-1}\left(1-\ln\left(\frac{1}{x+1}\right)\right)^{-(2\alpha+1)}, \quad x \geq 0, \quad (\alpha \geq 0), \quad (42)$$

15- For $\theta = 1, \alpha = 1, \beta = 1, c = 1$

$$f_{TL-CP}(x) = 2(x+1)^{-1}\left(1-\ln\left(\frac{1}{x+1}\right)\right)^{-3}, \quad x \geq 0, \quad (\beta \geq 0). \quad (43)$$

5. Some Related Distributions

It can be shown that the TL-CPIIG (θ, α, β, c) distribution is related with variable transformations to a wide range of well-known distributions such as TL-F, TL-Weibull (TL- exponential, exponential, TL-Rayleigh, Rayleigh), beta Type I (uniform), TL- beta Type II, TL- Pareto Type I, TL- Gompertz, TL-generalized logistic Type I, TL- extreme value, TL- Burr Type III, TL- compound Gompertz, TL-generalized uniform, TL- log beta Type I, TL- double truncated log beta Type II, TL- left truncated log beta Type I distributions. Table (1) summarizes the transformations from TL-CPIIG (θ, α, β, c) to other distributions. The proof in each case is straightforward.

Table (1) summarizes the above transformations applied to the TL-CPIIG (θ, α, β, c).

Table (1): Summary of Transformations Applied to the TL-CPIIG (θ, α, β, c) Distribution and Resulting Distribution.

Transformation	Distribution	Pdf	Range
$-\ln\left(\frac{c}{X+c}\right)$	$TL-F(\theta,\alpha,\beta)$	$\frac{2\alpha\theta}{\beta}\left(1-\frac{y_1}{\beta}\right)^{-(2\alpha+1)}\left(1-\left(1-\frac{y_1}{\beta}\right)^{-2\alpha}\right)^{\theta-1}$	$\boldsymbol{y_1 \geq 0}$, $(\boldsymbol{\theta,\alpha,\beta \geq 0})$
Special cases When $\theta = 1$ When $\theta = \beta = 1$	$F(\alpha,\beta)$ $F(\alpha)$	$\frac{2\alpha}{\beta}\left(1-\frac{y_1}{\beta}\right)^{-(2\alpha+1)}$ $2\alpha(1-y_1)^{-(2\alpha+1)}$	$\boldsymbol{y_1 \geq 0}, (\boldsymbol{\alpha,\beta \geq 0})$ $y_1 \geq 0, (\alpha \geq 0)$
$\boldsymbol{ln\left[\left(1-\frac{1}{\beta}ln\left(\frac{c}{X+c}\right)\right)\right]^{\frac{1}{d}}}$	TL- Weibull (θ,α,d)	$\boldsymbol{2\alpha\theta d y_2{}^{d-1} e^{-2\alpha y_2{}^d}\left(1-e^{-2\alpha y_2{}^d}\right)^{\theta-1}}$	$\boldsymbol{y_2 \geq 0}$, $(\boldsymbol{\theta,\alpha,d \geq 0})$
Special cases When $d = 1$ When $\theta = d = 1$ When $d = 2$ When $\theta = 1$ and $d = 2$	TL- exponential (θ,α) TL- exponential (α) TL- Rayleigh (θ,α) TL- Rayleigh (α)	$2\alpha\theta e^{-2\alpha y_2}(1-e^{-2\alpha y_2})^{\theta-1}$ $2\alpha e^{-2\alpha y_2}$ $4\alpha\theta y_2 e^{-2\alpha y_2{}^2}\left(1-e^{-2\alpha y_2{}^2}\right)^{\theta-1}$ $4\alpha y_2 e^{-2\alpha y_2{}^2}$	$y_2 \geq 0, (\theta,\alpha \geq 0)$ $y_2 \geq 0, (\alpha \geq 0)$ $y_2 \geq 0, (\theta,\alpha \geq 0)$ $y_2 \geq 0, (\alpha \geq 0)$
$\left(1-\frac{1}{\beta}\ln\left(\frac{c}{X+c}\right)\right)^{-1}$	beta Type I $(2\alpha,\theta)$	$2\alpha\theta y_3{}^{2\alpha-1}(1-y_3{}^{2\alpha})^{\theta-1}$	$0 < y_3 < 1$, $(\theta,\alpha \geq 0)$
Special cases When $\theta = 1$ When $\theta = 1$ and $\alpha = \frac{1}{2}$	beta Type I $(2\alpha, 1)$ uniform $(0,1)$	$2\alpha y_3{}^{2\alpha-1}$ 1	$0 < y_3 < 1$, $(\alpha \geq 0)$ $0 < y_3 < 1$, $\left(\alpha = \frac{1}{2}\right)$
$-\frac{1}{\beta}\ln\left(\frac{c}{X+c}\right)$	TL- beta Type II $(2\alpha,\theta)$	$2\alpha\theta(1+y_4)^{-(2\alpha+1)}(1-(1+y_4)^{-2\alpha})^{\theta-1}$	$y_4 > 0$, $(\theta,\alpha \geq 0)$
$\left(1-\frac{1}{\beta}\ln\left(\frac{c}{X+c}\right)\right)$	TL- Pareto Type I (θ,α)	$2\alpha\theta y_5{}^{-(2\alpha+1)}(1-y_5{}^{-2\alpha})^{\theta-1}$	$y_5 > 1$, $(\theta,\alpha \geq 0)$
$\ln\left[1+\ln\left(1-\frac{1}{\beta}\ln\left(\frac{c}{X+c}\right)\right)\right]^{\frac{1}{a}}$	TL- Gompertz (θ,α,a)	$2a\alpha\theta e^{-2\alpha(e^{ay_6}-1)+ay_6} \times\left(1-e^{-2\alpha(e^{ay_6}-1)}\right)^{\theta-1}$	$y_6 \geq 0$, $(\theta,\alpha,a \geq 0)$
$\ln\left[\beta\left(\ln\left(\frac{c}{X+c}\right)\right)^{-1}\right]$	TL-generalized logistic Type I (θ,α)	$2\alpha\theta e^{-y_7}(1-e^{-y_7})^{-(2\alpha+1)} \times(1-(1-e^{-y_7})^{-2\alpha})^{\theta-1}$	$,-\infty < y_7 < \infty$ $(\theta,\alpha \geq 0)$
$\ln\left[\ln\left(1-\frac{1}{\beta}\ln\left(\frac{c}{X+c}\right)\right)^{\sigma}\right]^{\frac{-1}{\sigma}}$	TL- extreme value (θ,α)	$2\alpha\theta e^{\frac{-2\alpha}{\sigma}e^{-\sigma y_8}-\sigma y_8}\left(1-e^{\frac{-2\alpha}{\sigma}e^{-\sigma y_8}}\right)^{\theta-1}$	$y_8 \geq 0, (\theta,\alpha \geq 0)$
$\frac{\alpha}{\beta}\ln\left(\frac{c}{X+c}\right)$	TL- Burr Type III (θ,α)	$2\theta\left(1-\frac{y_9}{\alpha}\right)^{-(2\alpha+1)} \times\left(1-\left(1-\frac{y_9}{\alpha}\right)^{-2\alpha}\right)^{\theta-1}$	$y_9 > 0$, $(\theta,\alpha \geq 0)$
$\ln\left(1-\ln\left(\frac{c}{X+c}\right)\right)$	TL- compound Gompertz (θ,α,β)	$\frac{2\alpha\theta}{\beta}e^{y_{10}}\left(1-\frac{1-e^{y_{10}}}{\beta}\right)^{-(2\alpha+1)} \times\left(1-\left(1-\frac{1-e^{y_{10}}}{\beta}\right)\right)^{\theta-1}$	$y_{10} \geq 0$, $(\theta,\alpha,\beta \geq 0)$

$\frac{1}{\beta}\left(1-\left(1-\frac{1}{\beta}\ln\left(\frac{c}{X+c}\right)\right)^{-1}\right)$	T -generalized uniform (θ, α, β)	$2\alpha\theta\beta(1-\beta y_{11})^{(2\alpha-1)} \times (1-(1-\beta y_{11})^{2\alpha})^{\theta-1}$	$y_{11} \geq 0$, $(\theta, \alpha, \beta \geq 0)$
$\frac{c}{X+c}$	TL- log beta Type I (θ, α, β)	$\frac{2\alpha\theta}{\beta}(y_{12})^{-1}\left(1-\frac{\ln(y_{12})}{\beta}\right)^{-(2\alpha+1)} \times\left(1-\left(1-\frac{\ln(y_{12})}{\beta}\right)^{-2\alpha}\right)^{\theta-1}$	$0 < y_{12} < 1$, $(\theta, \alpha, \beta \geq 0)$
$\frac{X+c}{c}$	TL- double truncated log beta Type II (θ, α, β)	$\frac{2\alpha\theta}{\beta}(y_{13})^{-1}\left(1+\frac{\ln(y_{13})}{\beta}\right)^{-(2\alpha+1)} \times\left(1-\left(1+\frac{\ln(y_{13})}{\beta}\right)^{-2\alpha}\right)^{\theta-1}$	$1 < y_{13} < \infty$, $(\theta, \alpha, \beta \geq 0)$
$\frac{1}{X+c}$	TL- left truncated log beta Type I (θ, α, β)	$\frac{\boldsymbol{2\alpha\theta}}{\boldsymbol{\beta}}(\boldsymbol{y_{14}})^{-1}\left(\boldsymbol{1}-\frac{\boldsymbol{ln(cy_{14})}}{\boldsymbol{\beta}}\right)^{-(2\alpha+1)} \times\left(\boldsymbol{1}-\left(\boldsymbol{1}-\frac{\boldsymbol{ln(cy_{14})}}{\boldsymbol{\beta}}\right)^{-2\alpha}\right)^{\theta-1}$	$0 < y_{14} < \frac{1}{c}$, $(\theta, \alpha, \beta \geq 0)$

6. Parameter Estimation

The maximum likelihood method is used for estimating the parameters of TL-CPIIG (θ, α,β, c).

Let $X = (x_1, x_2, \dots, x_n)$ be i. i. d random sample having pdf of (8), and then the likelihood function is given by:

$$L(x) = 2^n\theta^n\alpha^n\beta^{2n\alpha}\prod_{i=1}^{n}\left(\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha}\right)^{\theta-1}\right) \prod_{i=1}^{n}\left((x_i+c)^{-1}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-(2\alpha+1)}\right) \quad (44)$$

The natural logarithm of the likelihood function is:

$$\ell \equiv \ln L(x) = n(\ln 2+\ln\theta+\ln\alpha+2\alpha\ln\beta)+(\theta-1)\sum_{i=1}^{n}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha}\right) -\sum_{i=1}^{n}(x_i+c)-(2\alpha+1)\sum_{i=1}^{n}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right) \quad (45)$$

Differentiating (45) partially with respect to the parameters (θ, α, β, c) and equating the resulting derivatives to zero, we get the following maximum likelihood equations:

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta}+\sum_{i=1}^{n}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha}\right) = 0, \quad (46)$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha}+2n\ln\beta-2\ln\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)+\frac{2(\theta-1)\ln\left(\beta^{2\alpha+1}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha-1}\right)}{\left(\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha}\right)-1} = 0, \quad (47)$$

$$\frac{\partial\ell}{\partial\beta} = \frac{2n\alpha}{\beta}-\frac{2\alpha+1}{\beta-\ln\left(\frac{c}{x_i+c}\right)}-\frac{2\alpha(\theta-1)\left(\beta^{2\alpha-1}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{1-2\alpha}\right)\left(\beta\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{2}-\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-1}\right)}{\left(\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha}\right)-1} = 0, \quad (48)$$

$$\frac{\partial \ell}{\partial c} = -(x_i + c)^{-1} - (\theta - 1)\left[\frac{2\alpha\beta\left(\beta^{2\alpha-1}\left(\beta - \ln\left(\frac{c}{x_i + c}\right)\right)^{1-2\alpha}\right)(x_i + c)(c(x_i + c)^2 - (x_i + c)^{-1})}{c\left(\beta - \ln\left(\frac{c}{x_i + c}\right)\right)^2\left(\left(\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x_i + c}\right)\right)^{-2\alpha}\right) - 1\right)}\right] - \frac{(2\alpha + 1)(x_i + c)[c(x_i + c)^2 - (x_i + c)^{-1}]}{c\left(\beta - \ln\left(\frac{c}{x_i + c}\right)\right)} = 0, \quad (49)$$

In Section (6), we can obtain the estimates of unknown parameters by setting the last equations equal zero, but solving these equations simultaneously to get the unknown parameters $(\hat{\theta}, \hat{\alpha}, \hat{\beta}, \hat{c})$ in explicit form is mathematically complicated, so these estimates will be obtained numerically.

According to the invariance property of ML estimation, the MLE of any function $\xi(\vartheta)$ of θ is the function $\xi(\hat{\vartheta})$ of the MLE $\hat{\underline{\vartheta}}$ of $\underline{\vartheta}$. Since the equations (46-49) cannot closed form solution, they solved numerical simultaneously. So, the $\mathrm{MLE's}$ of the $r(x), h(x)$, (rhf) and $H_{TL}(x)$ are obtained by replacing the parameters θ, α, β and c (11), (12), (13) and (14) by the corresponding $\mathrm{MLE's}$. Hence, for given value of x, the MLE of $r(x)$ is given by:

$$\hat{R}_{TL-CP}(x) = 1 - \left(1 - \hat{\beta}^{2\hat{\alpha}}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x + \hat{c}}\right)\right)^{-2\hat{\alpha}}\right)^{\hat{\theta}}, \quad x \geq 0, \quad (50)$$

for given value of x, the MLE of $h(x)$ is given by

$$\hat{h}_{TL-CP}(x) = \frac{2\hat{\theta}\hat{\alpha}\hat{\beta}^{2\hat{\alpha}}(x + \hat{c})^{-1}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x + \hat{c}}\right)\right)^{-(2\hat{\alpha}+1)}\left(1 - \hat{\beta}^{2\hat{\alpha}}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x + \hat{c}}\right)\right)^{-2\hat{\alpha}}\right)^{\hat{\theta}-1}}{1 - \left(1 - \hat{\beta}^{2\hat{\alpha}}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x + \hat{c}}\right)\right)^{-2\hat{\alpha}}\right)^{\hat{\theta}}}; \quad x \geq 0, \quad (51)$$

for given value of x, the MLE of $rh(x)$ is given by:

$$\widehat{rh}_{TL-CP}(x) = \frac{2\hat{\theta}\hat{\alpha}\hat{\beta}^{2\hat{\alpha}}(x + \hat{c})^{-1}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x + \hat{c}}\right)\right)^{-(2\hat{\alpha}+1)}}{\left(1 - \hat{\beta}^{2\hat{\alpha}}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x + \hat{c}}\right)\right)^{-2\hat{\alpha}}\right)}; \quad x \geq 0, \quad (52)$$

and for given value of x, the MLE of $H_{TL}(x)$ is given by:

$$\hat{H}_{TL-CP}(x) = \ln\left(1-\hat{\beta}^{2\hat{\alpha}}\left(\hat{\beta}-\ln\left(\frac{\hat{c}}{x+\hat{c}}\right)\right)^{-2\hat{\alpha}}\right)^{\hat{\theta}}, \quad x \geq 0, \qquad (53)$$

where $\hat{\theta}$, $\hat{\alpha}$, $\hat{\beta}$ and $\hat{c}$ are the MLE of θ, α, β and c.

7. Simulation Study

In this section, a simulation is conducted to illustrate the performance of the presented ML estimates of the parameters based on complete samples. Moreover, confidence intervals (CIs) of the parameters. Illustrating is obtained results using Mathcad (14).

The steps of the procedure are as follows:

- For a given vector $\underline{\vartheta} = (\theta, \alpha, \beta, c)$, generate random samples of sizes (n = 30, 50, 100 and 150) from the TL-CPIIG (θ, α, β, c), using the following equation:
$$X = c\left(e^{-\beta\left(1-\left(1-u^{\frac{1}{\theta}}\right)^{\frac{-1}{2\alpha}}\right)}-1\right) \sim TL-CPIIG(\theta,\alpha,\beta,c).$$ [See Niyogi (2003)].
- Obtain numerically the MLE of θ, α, β and c by solving the equations $(46), (47), (48)$ and (49).
- The number of repetitions is m = 1000.
- Evaluate the performance of the estimates. To study the precision and variation of the estimates using the mean square error$(\mathrm{MSE}) = (\mathrm{Bias}^2) + \mathrm{variance}$.
- The estimates, bias (Bias), variance (Var), mean square error (MSE), interval estimation, upper, lower and length of the ML estimates where x = 2 are calculated. The computational results are displayed in following table (2), the initial values are $(\theta = 0.06,\ \alpha = 0.6,\ \beta = 0.03, c = 0.4)$.

Table (2) shows the ML of the estimates (ML), biases (Bias²), variance (Var), mean squared errors (MSEs), and interval estimation, upper, lower and length of the ML estimates of the parameters for each sample size.

Table (2): ML, Var, Bias, MSE and 95% confidence intervals of the MLE's of the TL-CPIIG (θ, α, β, c) for Different Sample Sizes n, Repetitions m =1000 and Initial values $(\theta = 0.06,\ \alpha = 0.6, \beta = 0.03, c = 0.4)$

N	Parameters	ML	Var	Bias	MSE	CIs		length
						U	L	
30	θ	2.147	0.198	0.417	0.220	3.020	1.274	1.746
	α	5.678	0.487	0.178	0.518	7.045	4.311	2.734
	β	0.120	0.001	0.020	0.002	0.192	0.048	0.144
	c	2.296	0.128	-0.204	0.170	2.998	1.595	1.403
50	θ	2.166	0.092	0.166	0.120	2.762	1.57	1.192
	α	5.516	0.931	0.016	0.932	7.408	3.624	3.784

	β	0.100	0.001	0.000	0.001	0.161	0.040	0.121
	c	2.465	0.197	-0.004	0.198	3.335	1.596	1.739
100	θ	2.030	0.033	0.030	0.034	2.386	1.674	0.712
	α	4.911	0.200	-0.589	0.547	5.787	4.035	1.752
	β	0.096	0.000	-0.004	0.000	0.130	0.062	0.067
	c	2.283	0.083	-0.217	0.130	2.847	1.720	1.127
150	θ	1.962	0.051	-0.038	0.052	2.403	1.521	0.882
	α	5.022	0.498	-0.478	0.727	6.406	3.638	2.768
	β	0.116	0.001	0.016	0.001	0.181	0.051	0.130
	c	2.119	0.102	-0.318	0.247	2.745	1.493	1.252

From table (2) it is noticed that as the sample size is increased the MSE is decreased and estimates are close to their actual values with small enough MSE.

8. Applications

In this section, we provide the application with real data sets to assess the flexibility of TL-CPIIG (θ, α, β, c) distribution.

We consider the criteria as AIC (Akaike information criterion), AICC (Akaike Information Corrected Criterion) and BIC (Bayesian Information Criterion), CAIC (Corrected Akaike Information Criterion) [See Hurvich and Tsai (1994)]. The better distribution is the one with the smallest values of the previous criteria, in the following, we considered two data sets:

Data Set 1

The first data set that represent 40 patients suffering from blood cancer (Leukemia) from one ministry of health hospital in Saudi Arabia. The ordered lifetime (in years) given as follows:

0.00315, 0.00496, 0.00616, 0.01145, 0.01208, 0.01263, 0.01414, 0.02025, 0.02036, 0.02162, 0.02211, 0.02370, 0.02532, 0.02693, 0.02805, 0.02910, 0.02912, 0.03192, 0.03263, 0.03348, 0.03348, 0.03427, 0.03499, 0.03534, 0.03767, 0.03751, 0.03858, 0.03986, 0.04049, 0.04244, 0.04323, 0.04381, 0.04392, 0.04397, 0.04647, 0.04753, 0.04929, 0.04973, 0.04381, 0.05074.

[See Atallah, Mahmoud, and Al-Zahrani (2014)].

Data Set 2

This data set consists of the waiting times (in seconds) percent, between 65 successive eruptions of the Kiama Blowhole. These values were record with the aid of digital watch on 12 July 1998 by Jim Irish and has been reference, the actual data are:

0.83, 0.51, 0.87, 0.60, 0.28, 0.95, 0.08, 0.27, 0.15, 0.10, 0.18, 0.16, 0.29, 0.54, 0.91, 0.08, 0.17, 0.55, 0.10, 0.35, 0.47, 0.77, 0.36, 0.17, 0.21, 0.36, 0.18, 0.40, 0.10, 0.07, 0.34, 0.27, 0.28, 0.56, 0.08, 0.25, 0.68, 1.46, 0.89, 0.18, 0.73, 0.69, 0.09, 0.37, 0.10, 0.82, 0.29, 0.08, 0.60, 0.61, 0.61, 0.18, 1.69, 0.25, 0.08, 0.26, 0.11, 0.83, 0.11, 0.42, 0.17, 0.12, 0.14, 0.09, 0.12

[See Pinho, Cordeiro and Nobre (2015) and Al-Saiary and Bakoban (2020)].

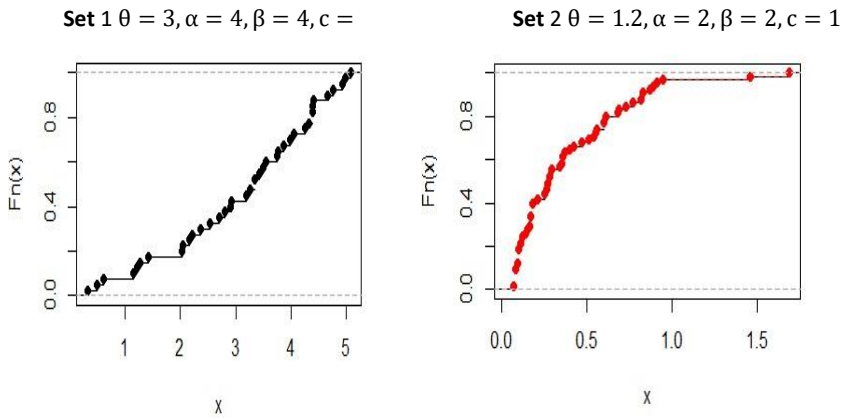

Figure 3: Plots of the Goodness of Fit of TL-CPIIG (θ, α, β, c) distribution using data set 1 and set 2.

Figure 3 shows that: at different values of the parameters, the curves of the fitted CDF represent monotone increasing functions in set 1for Leukemia data while, the curve of set 2for the waiting times data is monotone increasing functions then constant at $x \geq 1$.

Table (3): Statistics, 95% Confidence Interval of the Difference of the data sets, Akaike Information Criterion, Akaike Information Corrected Criterion, Bayesian Information Criterion and Repetitions m =1, Goodness of fit

Data		Set 1	Set 2
N		40	65
Mean		0.0312	0.3940
Mode		0.0335	0.0800
Minimum		0.0032	0.0700
Maximum		0.0508	1.6900
Range		0.0476	1.6200
Std. Deviation		0.0133	0.3366
Skewness		-0.5150	1.6010
Kurtosis		-0.6550	3.1540
Quantile	Q1	0.0218	0.1300
	Q2	0.0335	0.2800
	Q3	0.0430	0.6000
Confidence Intervals	U	0.0354	0.4774
	L	0.0269	0.3106
Length		0.0085	0.1668

AIC	-66.0750	538.3880
AICC	-64.9330	539.0540
BIC	80.4840	-523.1360
CAIC	-505.7530	0.0037
p-value	0.5727	0.1485

From table (3) it is noticed that as the sample size is increased the AIC, AICC and CAIC increased while BIC is decreased. The better distribution is the one with the smallest values of the previous criteria. Finally, two real data applications are analyzed to assess the flexibility of new model over existing distribution. One sample Kolmogorov-Smirnov (KS) test is applied on the two sets of data to test the goodness of fit. It is found that p-value is greater than $\alpha = 0.01$. So that the proposed model provides better results than the existing models.

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